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Journal of Pure and Applied Algebra 212 (2008) 1695–1715

JOURNAL OF PURE AND APPLIED ALGEBRA

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Algebraic connective K-theory and the niveau filtration

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Received 11 June 2007; received in revised form 31 October 2007
Available online 31 January 2008

Communicated by C.A. Weibel

Abstract

Algebraic connective K-theory is a universal homology theory that connects K-homology groups (in particular Chow groups) and algebraic K-theories in some intimate manner. In this report we investigate how certain groups in the Brown–Gersten–Quillen spectral sequence for quasi-projective schemes give rise to a definition of connective K-theory groups.

MSC: 14; 16

1. Introduction

In topology, there is the notion of connective K-theory which, in a way, unifies singular homology groups and topological K-groups.[8] Motivated by this lesson, we seek the corresponding notion of the algebraic connective K-theory in the context of algebraic geometry which connects Chow groups and Quillen K-groups.

The aim of this report is to establish a relatively simple definition of the algebraic connective K-theory for quasi-projective schemes of finite type over a field k. We shall demonstrate that the connective K-groups can be identified with the $D^2_{p,q}$ groups of the Brown–Gersten–Quillen spectral sequence which do constitute a homology theory for quasi-projective k-schemes. It will be shown that many properties of Chow groups and K-groups are shared by connective K-groups, including certain ring and module structures, Gysin morphisms, projective bundle theorem, and higher Chern classes for vector bundles. Finally we will turn to the connectivity issue and prove that Quillen K-groups are the localization of the connective K-groups at a distinguished element β while the quotient map by β can be viewed as the natural map from connective K-groups to K-homology groups; note that for appropriate bi-degrees, the K-homology groups are Chow groups. In particular, this quotient map is surjective onto Chow groups.

2. Definition of oriented Borel–Moore functors

Let k denote any field and $\mathbf{Sch}_k$ the category of schemes over k. The definition of an oriented Borel–Moore functor is inspired by Quillen's notion of an oriented homology theory [5].

A functor $H : \mathbf{Sch}_k \to \mathfrak{Ab}$ is called *additive* if for any finite family $\{X_1, \ldots, X_r\}$ of k-schemes, the homomorphism $\oplus_{i=1}^r H(X_i) \to H(\coprod_{i=1}^r X_i)$ induced by the inclusions $X_i \hookrightarrow \coprod_{i=1}^r X_i$ is an isomorphism. In particular, $H(\varnothing) = 0$.

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doi:10.1016/j.jpaa.2007.12.002

From now on we shall work in the category $\mathfrak{QP}_k$, the subcategory of $\mathbf{Sch}_k$ whose objects are quasi-projective schemes of finite type over k and morphisms are projective.

An *oriented Borel–Moore functor* on $\mathfrak{QP}_k$ is prescribed by the following conditions:

D1. A collection H of additive functors $H_{p,q} : \mathfrak{QP}_k \to \mathfrak{Ab}$ where $p, q \in \mathbb{Z}$ where $\mathfrak{Ab}$ is the category of Abelian groups;[1]

D2. For every flat morphism $f : Y \to X$ of relative dimension d with $X, Y \in \mathfrak{QP}_k$, there is a homomorphism of bi-graded Abelian groups $f^* : H_{p,q}(X) \to H_{p+d,q-d}(Y)$ for all $p, q \in \mathbb{Z}$;

together with the following axioms

A1. (Composition Law for Pull-backs) For any pair of composable flat morphisms $f : Y \to X$, $g : Z \to Y$ of relative dimensions d and e respectively, one has

$$(f \circ g)^* = g^* \circ f^* : H_{p,q}(X) \to H_{p+d+e,q-d-e}(Z),$$

and $Id_X^* = Id_{H_{*,*}(X)}$ for any $X \in \mathfrak{QP}_k$ and all $p, q \in \mathbb{Z}$.

A2. (Commutativity of Fibre-products) Let $f : Y \to X$ be proper, $g : Z \to Y$ flat of relative dimension r and the following diagram be a fibre-square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z, \end{array}$$

then $g^* f_* = f'_* g'^* : H_{p,q}(X) \to H_{p+r,q-r}(Y)$ for all integral pairs (p, q).

A3. (Localization Exact Sequence) Let $i : Z \hookrightarrow X$ be a closed embedding, and U be the open complement with immersion morphism $j : U \hookrightarrow X$. For such a triple (X, Z, U), there is a connecting homomorphism $\delta^H_{p+1,q} : H_{p+1,q}(U) \to H_{p,q}(Z)$ for all $p, q \in \mathbb{Z}$ giving rise to the following long exact sequences

$$\cdots \to H_{p+1,q}(X) \to H_{p+1,q}(U) \to H_{p,q}(Z) \to H_{p,q}(X) \to H_{p,q}(U) \to \cdots$$

which meet following naturality requirements.

(a) Given triples $Z \hookrightarrow X \hookleftarrow U, Z' \hookrightarrow X \hookleftarrow U'$ with embeddings $i : Z \hookrightarrow Z', i' : Z' \hookrightarrow X$ and $j' : U' \to U$, $j : U \to X$ the complementary open immersions, we have a commutative diagram

$$\begin{array}{ccccccc} \cdots \to H_{p+1,q}(U) & \xrightarrow{\delta} & H_{p,q}(Z) & \xrightarrow{i'_* \circ i_*} & H_{p,q}(X) & \xrightarrow{j^*} & H_{p,q}(U) \to \cdots \\ \downarrow{\scriptstyle j'^*} & & \downarrow{\scriptstyle i_*} & & \downarrow{=} & & \downarrow{\scriptstyle j'^*} \\ \cdots \to H_{p+1,q}(U') & \xrightarrow{\delta'} & H_{p,q}(Z') & \xrightarrow{i'_*} & H_{p,q}(X) & \xrightarrow{j'^* \circ j^*} & H_{p,q}(U') \to \cdots \end{array}$$

(b) Given a triples $Z \hookrightarrow X \hookleftarrow U$ and a flat morphism $f : Y \to X$ of relative dimension r, the following diagram commutes

$$\begin{array}{ccccccc} \cdots \to H_{p+1,q}(U) & \xrightarrow{\delta_X} & H_{p,q}(Z) & \xrightarrow{i_*} & H_{p,q}(X) & \xrightarrow{j^*} & \cdots \\ \downarrow{\scriptstyle f|_U^*} & & \downarrow{\scriptstyle f|_Z^*} & & \downarrow{\scriptstyle f^*} & & \\ \cdots \to H_{p+r+1,q-r}(f^{-1}(U)) & \xrightarrow{\delta_Y} & H_{p+r,q-r}(f^{-1}(Z)) & \xrightarrow{i'_*} & H_{p+r,q-r}(Y) & \xrightarrow{j'^*} & \cdots \end{array}$$

(c) Given a morphism between two triples, i.e.,

$$\begin{array}{ccccc} Z & \longrightarrow & X & \longleftarrow & U \\ \downarrow{\scriptstyle f|_Z} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f|_U} \\ Z' & \longrightarrow & Y & \longleftarrow & U' \end{array}$$

[1] Equivalently, we may consider a functor from $\mathfrak{QP}$to the category of bi-graded Abelian groups.

with f projective, the following diagram commutes

$$\begin{array}{ccccccc} \cdots \to H_{p+1,q}(U) & \xrightarrow{\delta} & H_{p,q}(Z) & \xrightarrow{i_*} & H_{p,q}(X) & \xrightarrow{j^*} & H_{p,q}(U) \to \cdots \\ \big\downarrow f|_{U*} & & \big\downarrow f|_{Z*} & & \big\downarrow f_* & & \big\downarrow f|_{U*} \\ \cdots \to H_{p+1,q}(U') & \xrightarrow{\delta'} & H_{p,q}(Z') & \xrightarrow{i'_*} & H_{p,q}(X) & \xrightarrow{j'^*} & H_{p,q}(U') \to \cdots \end{array}$$

A4. (Homotopy Invariance) Let $\pi : V \to X$ be a flat map between quasi-projective schemes whose fibres are affine spaces of dimension n, then $\pi^* : H_{p,q}(X) \to H_{p+n,q-n}(V)$ is an isomorphism for all integral pairs (p, q).

With the homotopy invariance property at hand, we can make the definition of the Chern class action of line bundles. Let L be a line bundle on X and denote by $\mathcal{L}$ the sheaf of sections of L. We define a homomorphism of bi-graded Abelian groups $c_1(\mathcal{L}) : H_{p,q}(X) \to H_{p-1,q+1}(X)$ as follows. Let $p : L \to X$ be the canonical projection and $s : X \to L$ be the zero section of L. Since p is flat, by homotopy equivalence p^* is an isomorphism. s is a closed embedding therefore proper. We assume

$$c_1(\mathcal{L}) = (p^*)^{-1} \cdot s_* : H_{p,q}(X) \to H_{p,q}(L) \to H_{p-1,q+1}(X)$$

and this is *the Chern class* of the line bundle L or $\mathcal{L}$.

Remark 2.1. Let L be a line bundle over X and $f : Y \to X$ be a morphism between quasi-projective schemes. The Chern class of L enjoys the following properties.

c1. In case that f is proper, we have $f_* \circ c_1(f^*\mathcal{L}) = c_1(\mathcal{L}) \circ f_*$.
c2. In case that f is flat of relative dimension d, we have $c_1(f^*\mathcal{L}) \circ f^* = f^* \circ c_1(\mathcal{L})$.
c3. Let L, M be two line bundles on X, then $c_1(\mathcal{L}) \circ c_1(\mathcal{M}) = c_1(\mathcal{M}) \circ c_1(\mathcal{L})$.

All these properties follow from the definition of Chern classes and functorial properties of pull-backs and push-forwards.

Remark 2.2. As a consequence of the additivity of $H_{p,q}$s and the localization axiom, we have the Mayer–Vietoris long exact sequences. Let U and V be open subschemes of X such that $U \cup V = X$, there are connecting homomorphisms $\partial_{p,q} : H_{p,q}(U \cap V) \to H_{p-1,q}(X)$ for all $p, q \in \mathbb{Z}$ making the following Mayer–Vietoris sequence

$$\cdots \to H_{p+1,q}(U \cap V) \to H_{p,q}(X) \to H_{p,q}(U) \oplus H_{p,q}(V) \to H_{p,q}(U \cap V) \to \cdots$$

a long exact sequence.

The morphism $\partial_{p,q}$ is constructed as follows. The localization sequence for $U \cap V$ in X yields a map $H_{p,q}(U \cap V) \to H_{p-1,q}(U^C \cup V^C) = H_{p-1,q}(U^C) \oplus H_{p-1,q}(V^C)$ which maps to $H_{p-1,q}(X) \oplus H_{p-1,q}(X)$ and whose image lies in the kernel of the natural morphism $H_{p-1,q}(X) \oplus H_{p-1,q}(X) \to H_{p-1,q}(X)$. The rest is diagram chasing.

3. The Brown–Gersten–Quillen spectral sequence

Let X be a quasi-projective k-scheme of finite type of dimension d.

We define $\mathcal{M}_p(X) = \{$coherent sheaves $\mathcal{F}$ on X whose support is a subscheme of dimension $\leqslant p$ on $X\}$. $\mathcal{M}_p(X) \subset \mathcal{M}(X)$ is a Serre subcategory, a nonempty full subcategory of $\mathcal{M}(X)$ which is closed under taking subobjects, quotients and extensions. There is a filtration on $\mathcal{M}(X)$:

$$0 = \mathcal{M}_0(X) \subset \mathcal{M}_1(X) \subset \cdots \subset \mathcal{M}_d(X) = \mathcal{M}(X).$$

Let X_p be the set of points of dimension p in X. There is an equivalence of categories between

$$\mathcal{M}_p(X)/\mathcal{M}_{p-1}(X) \cong \coprod_{x \in X_p} \mathcal{A}(\mathcal{O}_{x,X})$$

where $\mathcal{A}(\mathcal{O}_{x,X})$ is the category of $\mathcal{O}_{x,X}$-modules of finite length. Let $k(x)$ be the residue field of $\mathcal{O}_{x,X}$. By the dévisage theorem [6], we have $K_i(\mathcal{A}(O_{x,X})) \cong K_i(k(x))$ for all $i \in \mathbb{Z}$. These considerations assert that there are localization long exact sequences of K-groups

$$\begin{aligned}\cdots &\to K_i(\mathcal{M}_{p-1}(X)) \to K_i(\mathcal{M}_p(X)) \to \coprod_{x\in X_p} K_i(k(x)) \to K_{i-1}(\mathcal{M}_{p-1}(X)) \\ &\to K_{i-1}(\mathcal{M}_p(X)) \to \coprod_{x\in X_p} K_{i-1}(k(x)) \to K_{i-2}(\mathcal{M}_{p-1}(X)) \to \cdots\end{aligned}$$

for all integers p and i.

We denote

$$D^1_{p,q}(X) = K_{p+q}(\mathcal{M}_p(X)), \qquad E^1_{p,q}(X) = K_{p+q}(\mathcal{M}_p(X)/\mathcal{M}_{p-1}(X))$$

for all integers p and q. The previous discussion shows that

$$E^1_{p,q}(X) = \coprod_{x\in X_p} K_{p+q}(k(x))$$

and we may rewrite the above long exact sequences, replacing i by $p+q$, as

$$\begin{aligned}\cdots &\to D^1_{p-1,q+1}(X) \to D^1_{p,q}(X) \to E^1_{p,q}(X) \to D^1_{p-1,q}(X) \to D^1_{p,q-1}(X) \\ &\to E^1_{p,q-1}(X) \to D^1_{p-1,q-1}(X) \to \cdots\end{aligned} \tag{1}$$

for all integers p and q. This gives rise to the $n = 1$ level of the Brown–Gersten–Quillen Spectral Sequence. [1,9] There are naturally differentials

$$d^1_{p-1,q}(X) : E^1_{p,q}(X) \to E^1_{p-1,q}(X)$$

obtained by composing two of the arrows in (1): $E^1_{p,q}(X) \to D^1_{p,q}(X) \to E^1_{p-1,q}(X)$ which group the $E^1_{p,q}$ groups into complexes for fixed q. One can show that $d^1_{p,q}(X) : \coprod_{x\in X_p} K_{p+q}(k(x)) \to \coprod_{x\in X_{p-1}} K_{p+q-1}(k(x))$ are given by valuation maps of K-groups of fields and the resulting complexes are the *Gersten complexes* for X, denoted by $\mathfrak{G}_q(X)$:

$$\cdots \to \coprod_{x\in X_{p+1}} K_{p+q+1}(k(x)) \to \coprod_{x\in X_p} K_{p+q}(k(x)) \to \coprod_{x\in X_{p-1}} K_{p+q-1}(k(x)) \to \cdots .$$

The long exact sequences (1) engender another series of long exact sequences of Abelian groups which constitute the $n = 2$ page of the BGQ spectral sequence. The definition of $D^2_{p,q}(X)$ is given by

$$D^2_{p,q}(X) = \mathrm{Im}(D^1_{p-1,q+1}(X) \to D^1_{p,q}(X)),$$

whence there are three equivalent descriptions of $D^2_{p,q}(X)$:

1. $D^2_{p,q}(X) = \mathrm{Im}(K_{p+q}(\mathcal{M}_{p-1}(X)) \to K_{p+q}(\mathcal{M}_p(X)))$;
2. The cokernel of the map $E^1_{p,q+1}(X) \to D^1_{p-1,q+1}(X)$, i.e.,

   $$\coprod_{x\in X_p} K_{p+q+1}(k(x)) \to K_{p+q}(\mathcal{M}_{p-1}(X)) \to D^2_{p,q}(X) \to 0$$

   is exact.
3. The kernel of the map $D^1_{p,q}(X) \to E^1_{p,q}(X)$, i.e,

   $$0 \to D^2_{p,q}(X) \to K_{p+q}(\mathcal{M}_p(X)) \to \coprod_{x\in X_p} K_{p+q}(k(x))$$

   is exact.

Also according to the definition,

$$E^2_{p,q}(X) = \ker d^1_{p,q}(X)/\mathrm{im}\, d^1_{p+1,q}(X)$$

where $d^1_{p,q}(X)$ is the differential map addressed above. That is, the $E^2_{p,q}(X)$ groups are the cohomology groups of the Gersten complexes for fixed q.

An important example is the case when $q = -p$ and X is a finite type k-scheme. There, we are looking at the cohomology group of

$$\coprod_{x\in X_{p+1}} K_1(k(x)) \xrightarrow{d^1_{p,-p}(X)} \coprod_{x\in X_p} K_0(k(x)) \longrightarrow \coprod_{x\in X_{p-1}} K_{-1}(k(x)),$$

or,

$$\coprod_{x\in X_{p+1}} k(x)^\times \to \coprod_{x\in X_p} \mathbb{Z} \to 0.$$

Unraveling the definition of the differential $d^1_{p,-p}(X)$, one concludes that its image coincides with dimension p cycles that are rationally equivalent to 0 (e.g., c.f. [9]). Hence

$$E^2_{p,-p}(X) = CH_p(X).$$

The $n = 2$ level long exact sequences of the BGQ Spectral Sequence relate the $D^2(X)$ and $E^2(X)$ groups and say that for all $p, q \in \mathbb{Z}$, the following sequences are exact:

$$\cdots \to D^2_{p,q}(X) \to D^2_{p+1,q-1}(X) \to E^2_{p,q}(X) \to D^2_{p-1,q}(X) \to D^2_{p,q-1}(X) \to \cdots. \tag{2}$$

Let us now turn to the limit of the BCG spectral sequence. Consider the direct system

$$\cdots \to D^1_{p-1,q+1}(X) \to D^1_{p,q}(X) \to D^1_{p+1,q-1}(X) \to \cdots$$

whose direct limit is denoted by $D_{p+q}(X)$. There are the attendant groups

$$F_p D_{p+q}(X) = \mathrm{im}\ (D^1_{p,q}(X) \to D_{p+q}(X))$$

and $0 \subset F_0 D_{p+q}(X) \subset F_1 D_{p+q}(X) \subset \cdots \subset F_p D_{p+q}(X) \subset \cdots$ gives a canonical filtration on $D_{p+q}(X)$. In more concrete terms, since for $p > d = \dim(X)$, $D^1_{p,q}(X) = K_{p+q}(\mathcal{M}_p(X))$, the abutment terms of the BGQ spectral sequence $D_{p+q}(X) = K_{p+q}(\mathcal{M}(X))$ and we obtain the niveau or topological filtration on $K_{p+q}(\mathcal{M}(X))$ given by

$$F_p K_{p+q}(\mathcal{M}(X)) = \mathrm{im}\,(K_{p+q}(\mathcal{M}_p(X)) \to K_{p+q}(\mathcal{M}(X)))$$

for all $p, q \in \mathbb{Z}$. Since for $p < 0$, $D^1_{p,q}(X) = K_{p+q}(\mathcal{M}_p(X)) = 0$, standard analysis of spectral sequences says the BGQ spectral sequence is strongly convergent, that is, for n large, the BGQ spectral sequences stabilize and

$$E^n_{p,q}(X) \cong E^{n+1}_{p,q}(X) \cong \cdots \cong E^\infty_{p,q}(X) \cong F_p D_{p+q}(X)/F_{p-1} D_{p+q}(X),$$

or notationally

$$E^1_{p,q}(X) = \coprod_{x\in X_p} K_{p+q}(k(x)) \Longrightarrow D_{p+q}(X) = K_{p+q}(\mathcal{M}(X))$$

for all $p, q \in \mathbb{Z}$. Consequently, one may regard the BGQ spectral sequences as relating the K-groups of points on the scheme to those of the scheme. But they in fact carry more information.

4. Oriented Borel–Moore functors in the BGQ spectral sequences

4.1. Functorial properties of $D^1_{p,q}$ and $E^1_{p,q}$ groups

All the $D^1_{p,q}(X)$ and $E^1_{p,q}(X)$ groups have the right functorial properties for any quasi-projective scheme X.

Proposition 4.1. *Let $f : X \to Y$ be a proper morphism between two quasi-projective schemes, we have push-forwards $f_* : D^1_{p,q}(X) \to D^1_{p,q}(Y)$.*

Proof of the Proposition. This is the same proof of Quillen [6]. We have a full subcategory $F_p(X, f)$ in $M_p(X)$ on which the restriction of f_* is exact. Since X is quasi-projective, every object in $M_p(X)$ admits a finite resolution whose successive quotients lie in $F_p(X, f)$ and hence the embedding of $F_p(X, f) \hookrightarrow M_p(X)$ induces a homotopy equivalence on the geometric realizations of both categories. □

It is plain from definitions that push-forwards on $M_p(X)$ commute with $M_{p-1}(X) \hookrightarrow M_p(X)$ and induce a push-forward on $M_p(X) \to M_p(X)/M_{p-1}(X)$.

Corollary 4.1. *Such an f induces a push-forward $f_* : E^1_{p,q}(X) \to E^1_{p,q}(Y)$, and the f_* s are functorial on the long exact sequences (1).*

Again following Quillen, one shows that push-forwards obey the composition law. For two composable proper morphisms f and g, $(g \circ f)_* = g_* f_*$.

Given a flat morphism $f : Y \to X$ of relative codimension r, since for $\mathcal{F} \in \mathcal{M}_p(X)$, $\dim(f^{-1}(\text{support}(\mathcal{F}))) = \dim(\text{support}(f^*\mathcal{F})) \leqslant p + r$, there are exact pull-back functors $f^* : \mathcal{M}_p(X) \to \mathcal{M}_{p+r}(Y)$ for all integer p giving rise to pull-back on the corresponding K-groups $f^* : D^1_{p,q}(X) \to D^1_{p+r,q-r}(Y)$. Since pull-backs commute with $M_{p-1}(X) \hookrightarrow M_p(X)$, one shows that a flat f induces $f^* : E^1_{p,q}(X) \to E^1_{p+r,q-r}(Y)$. Let us summarize.

Proposition 4.2. *A flat morphism $f : Y \to X$ of relative dimension r induces pull-backs $f^* : D^1_{p,q}(X) \to D^1_{p+r,q-r}(Y)$ and $f^* : E^1_{p,q}(X) \to E^1_{p+r,q-r}(Y)$ and they are functorial on the long exact sequences (1).*

It is straightforward to check that the composition law holds for pull-backs on $D^1_{p,q}$ and $E^1_{p,q}$ groups.

From the definitions of $D^2_{p,q}$ and $E^2_{p,q}$ groups, it is evident that both push-forwards and pull-backs are bequeathed unto these groups and are functorial on the long exact sequences (2).

Finally, let us remark that from definitions, $D^1_{p,q}$ and $E^1_{p,q}$ groups are additive (e.g., $D^1_{p,q}(\coprod_i X_i) \cong \coprod_i D^1_{p,q}(X_i)$ and these isomorphisms commute with (1)), and so are $D^2_{p,q}$ and $E^2_{p,q}$ groups.

4.2. The K-theory

Let X be quasi-projective over k. we denote by $K_i(X)$ the ith Quillen K-group of X and $K(X) = \coprod_i K_i(X)$ its total K-group. There is a product structure on $K(X)$ turning $K(X)$ into a graded ring [10]. Therefore there is the bi-graded ring $K(X)[\beta]$ where β is an indeterminant. Specifically, $K_i(X)\beta^m \times K_j(X)\beta^n \to K_{i+j}(X)\beta^{m+n}$ for all integers i, j, m and n. We define

$$K_{p,q}(X) = K_{p+q}(X)\beta^p$$

and the total group $K_T(X) = \coprod_{p,q\in\mathbb{Z}} K_{p,q}(X)$ is isomorphic to $K(X)[\beta, \beta^{-1}]$. It follows that β is identified as an element of $K_{1,-1}(X)$. This is the algebraic K-theory of X. That the $K_{p,q}$ groups form oriented Borel–Moore functors was established by Quillen in [6].

- D1. The $K_{p,q}$s are additive functors. Given a proper morphism between quasi-projective schemes $f : X \to Y$, there is an induced morphism $f_* : K_{p+q}(X) \to K_{p+q}(Y)$ hence $f_* : K_{p,q}(X) = K_{p+q}(X)\beta^p \to K_{p,q}(Y) = K_{p+q}(Y)\beta^p$ for all integers p and q. The additivity of the $K_{p,q}(X)$ groups follows from that of the $K_{p+q}(X)$ groups.

- D2. The pull-backs exist. Given an flat morphism between quasi-projective schemes $f : Y \to X$ of relative dimension r, there is an induced $f^* : K_{p+q}(X) \to K_{p+q}(Y)$ which gives rise to a map $f^* : K_{p,q}(X) = K_{p+q}(X)\beta^p \to K_{p+r,q-r}(Y) = K_{p+q}(Y)\beta^{p+r}$ for all integers p and q.
- A1. Composition law for pull-backs. Clear from that of the Quillen K-groups.
- A2. Commutativity of fibre-products. For a Cartesian diagram as in Axiom 2, Quillen showed [6](Proposition 2.11) that the functorial properties of the pull-backs and push-forwards imply $g^* \cdot f_* = f'_* \cdot g'^* : K_{p+q}(X) \to K_{p+q}(Y)$ and consequently $g^* \cdot f_* = f'_* \cdot g'^* : K_{p,q}(X) = K_{p+q}(X)\beta^p \to K_{p+r,q-r}(Y) = K_{p+q}(Y)\beta^{p+r}$ for all integers p and q.
- A3. Localization exact sequence. Let (X, Z, U) be a triple as given in the settings of Axiom 3. There is the localization exact sequence of the Quillen K-groups $\cdots \to K_{i+1}(U) \to K_i(Z) \to K_i(X) \to K_i(U) \to K_{i-1}(Z) \to \cdots$ satisfying the naturality properties as in Axiom 3. Hence we obtain the long exact sequence $\cdots \to K_{p+q+1}(U)\beta^{p+1} \to K_{p+q}(Z)\beta^p \to K_{p+q}(X)\beta^p \to K_{p+q}(U)\beta^p \to \cdots$, or,
$$\cdots \to K_{p+1,q}(U) \to K_{p,q}(Z) \to K_{p,q}(X) \to K_{p,q}(U) \to K_{p-1,q}(Z) \to \ldots$$
for all integers p and q which also meets the naturality requirements.
- A4. Homotopy Invariance. Given $\pi : V \to X$, a flat map between quasi-projective schemes whose fibres are affine spaces of dimension r, $\pi^* : K_n(X) \to K_n(V)$ is an isomorphism for any integer n, and so is $\pi^* : K_{p,q}(X) = K_{p+q}(X)\beta^p \to K_{p+r,q-r}(V) = K_{p+q}(V)\beta^{p+r}$.

4.3. The K-homology

For a quasi-projective scheme X and integers p and q, we define the (p, q)th K-homology group $A_{p,q}(X)$ to be
$$A_{p,q}(X) = E^2_{p,q}(X),$$
and $A(X) = \coprod_{p,q\in\mathbb{Z}} A_{p,q}(X)$ is the total group. We have seen for $p = -q$, $A_{p,-p}(X) = E^2_{p,-p}(X) = CH_p(X)$ where $CH_p(X)$ is the pth Chow group of X.

The verification that the $A_{p,q}$ groups form an Oriented Borel–Moore functor hinges on the fact that the $D^2_{p,q}$ groups also do the same. We shall assume that result in this part of the discussion of the $A_{p,q}$s and examine the $D^2_{p,q}$ groups in Section 5.1.

D1. The $A_{p,g}$s are additive functors.

D2. The pull-backs exist for the $A_{p,q}$s.

A1. Composition law for the pull-backs.

All these were established in 4.1.

A2. Commutativity of fibre-products. In Section 5.1, we shall prove $g^* \cdot f_* = f'_* \cdot g'^* : D^2_{p,q}(X) \to D^2_{p+d,q-d}(Y)$ for a cartesian square as in Axiom 2. The equality for $E^2_{p,q}$ now follows from (2) and the five-lemma.

A3. Localization exact sequences. Let (X, Z, U) be a triple as in the settings of A3. Since $E^1_{p,q}(X) = \coprod_{x\in X_p} K_{p+q}(k(x))$ and $X_p = Z_p \coprod U_p$, $\mathfrak{G}_q(Z)$ is a subcomplex of $\mathfrak{G}_q(X)$ and in fact
$$0 \to \mathfrak{G}_q(Z) \to \mathfrak{G}_q(X) \to \mathfrak{G}_q(U) \to 0$$
is exact. The localization exact sequence with fixed q is the cohomology long exact sequence of this short exact sequence of Gersten complexes. The functorial properties of the Gersten complexes and the above short exact sequences ensure the naturality requirements for the localization exact sequences.

A4. Homotopy Invariance. Let $\pi : V \longrightarrow X$ be a flat map between quasi-projective schemes whose fibres are affine spaces of dimension r, in Section 5.1 we shall establish that $\pi^* : D^2_{p,q}(X) \to D^2_{p+r,q-r}(V)$ is an isomorphism for any integral pair (p, q). Now the result for $A_{p,q}$ groups is attained on account of (2) and the five-lemma.

The main focus of this report is the $D^2_{p,q}$ groups, we will show that they form oriented Borel–Moore functors in the next section.

5. The CK-groups

5.1. Definition of CK-groups

Let X be quasi-projective, we define the *CK-homology groups* or the *algebraic connective K-groups*

$$CK_{p,q}(X) = D^2_{p+1,q-1}(X)$$

and denote $CK(X) = \coprod_{p,q\in\mathbb{Z}} CK_{p,q}(X)$. That these groups indeed constitute oriented Borel–Moore functors shall be the content of the remainder of this section.

Recall that we have three equivalent descriptions of $CK_{p,q}(X)$

1. $CK_{p,q}(X) = \mathrm{Im}(K_{p+q}(\mathcal{M}_p(X)) \to K_{p+q}(\mathcal{M}_{p+1}(X)))$
2. $\coprod_{x\in X_{p+1}} K_{p+q+1}(k(x)) \to K_{p+q}(\mathcal{M}_p(X)) \to CK_{p,q}(X) \to 0$
3. $0 \to CK_{p,q}(X) \to K_{p+q}(\mathcal{M}_{p+1}(X)) \to \coprod_{x\in X_{p+1}} K_{p+q}(k(x))$.

Rewriting (2), we obtain the following long exact sequence

$$\cdots \to CK_{p-1,q+1}(X) \to CK_{p,q}(X) \to A_{p,q}(X) \to CK_{p-2,q+1}(X) \to \cdots . \tag{3}$$

In 4.1, we saw that $CK_{p,q}$ is a additive functor on $\mathbf{QP}_k$ which admits pull-backs that are functorial for equi-dimensional flat morphisms, i.e., Definitions 1 and 2 and Axiom 1 of an oriented Borel–Moore functor are satisfied.

Given a Cartesian square as in Axiom 2, we need to show: $g^* \cdot f_* = f'_* \cdot g'^* : CK_{p,q}(X) \to CK_{p+r,q-r}(Y)$. Since the diagram is Cartesian, $g^* \cdot f_* = f'_* \cdot g'^* : \mathcal{M}_p(X) \to \mathcal{M}_{p+d}(Y)$. As noted previously, the embedding $F_p(X, f) \hookrightarrow \mathcal{M}_p(X)$ is a homotopy equivalence. We first show $g'^* : F_p(X, f) \to F_{p+d}(W, f')$. Let $\mathcal{F} \in F_p(X, f)$, then $R^i f_*(\mathcal{F}) = 0$ for $i > 0$. Since g'^*, g^* are exact functors, we have $R^i f'_*(g'^*\mathcal{F}) = R^i(f'_* \cdot g'^*)(\mathcal{F}) = R^i(g^* \cdot f_*)(\mathcal{F}) = g^* R^i f_*(\mathcal{F}) = 0$, i.e., $g'^*\mathcal{F} \in F_{p+d}(W, f')$. Therefore we have the commutative

$$\begin{tikzcd}
F_{p+d}(W, f') \arrow[d, "f'_*"] & F_p(X, f) \arrow[l, "g'^*"'] \arrow[d, "f_*"] \\
\mathcal{M}_{p+d}(Y) & \mathcal{M}_p(Z) \arrow[l, "g^*"']
\end{tikzcd}$$

in which all arrows are exact. This induces $g^* \cdot f_* = f'_* \cdot g'^* : K_{p+q}(\mathcal{M}_p(X)) \to K_{p+q}(\mathcal{M}_{p+d}(Y))$. All the maps involved commute with maps in (1). By the definition of $CK_{p,q}$, the desired result is established.

Theorem 5.1. *Let (X, Z, U) be a triple as in Axiom 3, there is a connecting homomorphism $\delta^{CK}_{p+1,q} : CK_{p+1,q}(U) \to CK_{p,q}(Z)$ for all $p, q \in \mathbb{Z}$ giving rise to the following long exact sequence:*

$$\cdots \to CK_{p+1,q}(X) \to CK_{p+1,q}(U) \to CK_{p,q}(Z) \to CK_{p,q}(X) \cdots .$$

Proof. $\mathcal{M}_{p+1}(Z)$ is the full subcategory of $\mathcal{M}_{p+1}(X)$ whose objects are annihilated by the ideal sheaf of Z. Let $\mathcal{B}_{p+1} \subset \mathcal{M}_{p+1}(X)$ be the Serre subcategory of sheaves $\mathcal{F}$ with $\mathcal{F}|_U = 0$. $\mathcal{M}_{p+1}(Z) \subset \mathcal{B}_{p+1}$ is a homotopy equivalence by dévissage, and $\mathcal{M}_{p+1}(X)/\mathcal{B}_{p+1}$ is equivalent to $\mathcal{M}_{p+1}(U)$ [3]. Localization theorem of K-groups says that $\cdots \to K_{i+1}(\mathcal{M}_{p+1}(U)) \to K_i(\mathcal{M}_{p+1}(Z)) \to K_i(\mathcal{M}_{p+1}(X)) \to K_i(\mathcal{M}_{p+1}(U)) \to \cdots$ is exact.

Consider

$$\begin{tikzcd}
 & \coprod_{x\in X_{p+2}} K_{i+2}(k(x)) \arrow[r, "g^X_{i+2}"] \arrow[d, "p_{i+2}"] & K_{i+1}(\mathcal{M}_{p+1}(X)) \arrow[r] \arrow[d, "j^K_{i+1}"] & CK_{p+1,q}(X) \arrow[r] \arrow[d] & 0 \\
 & \coprod_{x\in U_{p+2}} K_{i+2}(k(x)) \arrow[r, "g^U_{i+2}"] & K_{i+1}(\mathcal{M}_{p+1}(U)) \arrow[r] \arrow[d, "\delta^K_{i+1}"'] & CK_{p+1,q}(U) \arrow[r] & 0 \\
0 \arrow[r] & CK_{p,q}(Z) \arrow[r, "f^Z_i"] \arrow[d] & K_i(\mathcal{M}_{p+1}(Z)) \arrow[r, "\delta^Z_i"] \arrow[d, "i^K_i"] & \coprod_{x\in Z_{p+2}} K_i(k(x)) \arrow[d, "i_i"] & \\
0 \arrow[r] & CK_{p,q}(X) \arrow[r] \arrow[d] & K_i(\mathcal{M}_{p+1}(X)) \arrow[r, "\delta^X_i"] \arrow[d] & \coprod_{x\in X_{p+2}} K_i(k(x)) \arrow[d] & \\
0 \arrow[r] & CK_{p,q}(U) \arrow[r] & K_i(\mathcal{M}_{p+1}(U)) \arrow[r] & \coprod_{x\in U_{p+2}} K_i(k(x)) &
\end{tikzcd}$$

where $i = p + q$. We define the connecting homomorphism $\delta^{CK}_{p+1} : CK_{p+1,q}(U) \to CK_{p,q}(Z)$ as follows. An element $b \in CK_{p+1,q}(U)$ lifts to $b' \in K_{i+1}(\mathcal{M}_{p+1}(U))$ whose image under δ^K_{i+1} is $b'' \in K_i(\mathcal{M}_{p+1}(Z))$. By commutativity of the diagram, $i_i \circ \delta^Z_i(b'') = \delta^X_i \circ i^K_i(b'') = \delta^X_i \circ i^K_i \circ \delta^K_{i+1}(b') = 0$. Since i_i in an injection, we conclude $\delta^Z_i(b'') = 0$. Hence $b'' = f^Z_i(c)$ for a unique $c \in CK_{p,q}(Z)$ (i.e., the map δ^K_{i+1} factors through $CK_{p,q}(Z)$). We stipulate that $\delta^{CK}_{p+1}(b) = c$. Given another lift b'_1 of b in $K_{i+1}(\mathcal{M}_{p+1}(U))$, the difference $b' - b'_1 = g^U_{i+2}(d)$ for some $d \in \coprod_{x\in U_{p+2}} K_{i+2}(k(x))$. Since p_{i+2} is surjective, $d = p_{i+2}(d')$ for some $d' \in \coprod_{x\in X_{p+2}} K_{i+2}(k(x))$, and $\delta^K_{i+1} \circ g^U_{i+2}(d) = \delta^K_{i+1} \circ g^U_{i+2} \circ p_{i+2}(d') = \delta^K_{i+1} \circ j^K_{i+1} \circ g^X_{i+2}(d') = 0$ by virtue of the commutativity of the square and the exactness of the middle column. So b' and b'_1 give the same b'' wherefore δ^{CK}_{p+1} is well defined. Now let us verify the exactness at different joints.

- $CK_{p,q}(Z) \to CK_{p,q}(X) \to CK_{p,q}(U)$ is exact. This is clear.
- $CK_{p+1,q}(U) \to CK_{p,q}(Z) \to CK_{p,q}(X)$ is exact. This is transparent in virtue of the definition of δ^{CK}_{p+1} and the exactness of the middle column.
- $CK_{p+1,q}(X) \to CK_{p+1,q}(U) \to CK_{p,q}(Z)$ is exact. This is again evident from the definition of δ^D_{p+1}, that δ^K_{i+1} factors through $CK_{p,q}(Z)$, and the exactness of the middle column. ■

Theorem 5.2. *The localization sequences of the CK-homology theory satisfy the naturality conditions.*

Proof. (a). The commutativity of squares involving i_* and j^* respectively are consequences of the functorial properties of push-forwards and pull-backs. To see the commutativity of a square involving the connecting morphisms, observe that in defining $\delta'^{CK} : CK_{p+1,q}(U') \to CK_{p,q}(Z')$ for the triple (X, Z', U'), one creates a diagram similar to the one for (X, Z, U). Each term in the old diagram maps to the corresponding term in the new diagram and all squares are commutative. Those that involve push-forwards and pull-backs are again commutative by functoriality of such maps; the ones containing the δ^Ks are commutative from the naturality of the localization sequence of K-theory [9] (p. 42, 4.9; p. 61, 5.16). Hence we arrive at the commutativity of the square involving δ^{CK}.

The proofs of (b) and (c) are similar. ■

This completes the verification of Axiom 3.

Before proceeding to the verification of Axiom 4, we compute the connective K-groups for a field. Let k be a field.

Proposition 5.1.

$$CK_{p,q}(k) \cong \begin{cases} K_{p+q}(k) & \text{for } p \geqslant 0 \\ 0 & \text{for } p < 0. \end{cases}$$

Proof. From the 2nd description (3) of $CK_{p,q}$, we have:

$$\coprod_{x \in \mathrm{spec}(k)_{p+1}} K_{p+q+1}(k(x)) \to K_{p+q}(\mathcal{M}_p(k)) \to CK_{p,q}(k) \to 0$$

- $p \geqslant 0$. $\mathrm{spec}(k)_p = \varnothing$, and $\mathcal{M}_p(k) = \mathcal{M}(k)$, hence $CK_{p,q}(k) \cong K_{p+q}(k)$.
- $p < 0$. $\mathcal{M}_p(k) = 0$, hence $CK_{p,q}(k) = 0$. ■

Corollary 5.1. *Let $\mathbb{A}^1 (= \mathbb{A}^1_k)$ be the affine line over k.*

$$CK_{p,q}(\mathbb{A}^1) \cong \begin{cases} K_{p+q}(k) & \text{for } p > 0 \\ 0 & \text{for } p \leqslant 0 \end{cases}$$

and $\pi : \mathbb{A}^1 \to \mathrm{spec}(k)$ induces an isomorphism $\pi^* : CK_{p,q}(k) \cong CK_{p+1,q-1}(\mathbb{A}^1)$.

Proof. Examine

$$\coprod_{x \in \mathbb{A}^1{}_{p+1}} K_{p+q+1}(k(x)) \to K_{p+q}(\mathcal{M}_p(\mathbb{A}^1)) \to CK_{p,q}(\mathbb{A}^1) \to 0 \tag{4}$$

- $p > 0$. $\mathbb{A}^1_{p+1} = \varnothing$, and $\mathcal{M}_p(\mathbb{A}^1) = \mathcal{M}(\mathbb{A}^1)$, so $CK_{p,q}(\mathbb{A}^1) \cong K_{p+q}(\mathcal{M}_p(\mathbb{A}^1)) = K_{p+q}(\mathcal{M}(\mathbb{A}^1)) = K_{p+q}(\mathbb{A}^1) \cong K_{p+q}(k) \cong CK_{p-1,q+1}(k)$ where the penultimate isomorphism is given by π^*.
- $p = 0$. (1) says $0 \to K_q(\mathcal{M}_0(\mathbb{A}^1)) \to \coprod_{x \in \mathbb{A}^1{}_0} K_q(k(x)) \to 0$ is exact while the only point of dimension 1 in $\mathbb{A}^1$ is the generic point. Therefore (4) reads

  $$K_{q+1}(k(t)) \to \coprod_{x \in \mathbb{A}^1{}_0} K_q(k(x)) \to CK_{0,q}(\mathbb{A}^1) \to 0$$

  where $k(t)$ is the function field of $\mathbb{A}^1$ and the first arrow is none other than $d^1_{0,q}(\mathbb{A}^1)$. The localization exact sequence for K-theory says $\cdots \to K_{q+1}(Z) \to K_{q+1}(\mathbb{A}^1) \to K_{q+1}(U) \to K_q(Z) \to \cdots$ is exact for any triple (X, Z, U) as in A3. Taking the direct limit of all closed subscheme $Z \hookrightarrow X$ and on account of the functorial property of localization sequences and the exactness of direct limit, we get

  $$\cdots \to \varinjlim K_{q+1}(Z) \to K_{q+1}(\mathbb{A}^1) \to \varinjlim K_{q+1}(U) \to \varinjlim K_q(Z) \to \cdots$$

  or

  $$\cdots \to \coprod_{x \in \mathbb{A}^1{}_0} K_{q+1}(k(x)) \to K_{q+1}(k) \to K_{q+1}(k(t)) \to \coprod_{x \in \mathbb{A}^1{}_0} K_q(k(x)) \to \cdots$$

  where the penultimate arrow is identified as $d^1_{0,q}(\mathbb{A}^1)$ and the middle arrow $\iota : K_{q+1}(k) \to K_{q+1}(k(t))$ is induced by $\mathrm{spec}(k(t)) \to \mathbb{A}^1 \to \mathrm{spec}(k)$ which is the structure morphism. It is known that ι is a split injection. It follows that $d^1_{0,q}(\mathbb{A}^1)$ is a surjection and consequently $CK_{0,q}(\mathbb{A}^1) = 0$.
- $p < 0$. $\mathcal{M}_{p-1}(\mathbb{A}^1) = \varnothing$, and $K_{p+q}(\mathcal{M}_{p-1}(\mathbb{A}^1)) = 0$. Therefore $CK_{p,q}(\mathbb{A}^1) = 0$. ■

Lemma 5.1. *Let $Z \to X$ be a closed subscheme such that $\mathcal{I}_Z$ is nilpotent, then $CK_{p,q}(Z) \cong CK_{p,q}(X)$.*

Proof of the Lemma. Since $\mathcal{I}_Z$ is nilpotent, $U = Z^C = \varnothing$ and the localization sequence dictates that $CK_{p,q}(Z) \cong CK_{p,q}(X)$. □

Theorem 5.3. *For a flat morphism $\pi : V \to X$ whose fibres are affine spaces of dimension n, $\pi^* : CK_{p,q}(X) \to CK_{p+n,q-n}(V)$ is an isomorphism for all pairs (p, q).*

Proof. Flatness and affine fibres are stable under base extensions. let $Z \hookrightarrow X$ be a closed subscheme and U the open complement, the following diagram is commutative

$$\begin{array}{ccccccc} \cdots \to CK_{p,q}(Z) & \longrightarrow & CK_{p,q}(X) & \longrightarrow & CK_{p,q}(U) & \longrightarrow & CK_{p-1,q}(Z) \to \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to CK_{p+n,q-n}(V_Z) & \longrightarrow & CK_{p,q}(V) & \longrightarrow & CK_{p+n,q-n}(V_U) & \longrightarrow & CK_{p+n-1,q-n}(V_Z) \to \cdots \end{array}$$

By the five-lemma, the conclusion holds for any of the X, Z, U if it is known to be true for the other two. Using Noetherian induction, we may assume all proper closed subschemes $Z \subset X$ subject to the proposition. The preceding lemma allows us to reduce to the case that X is reduced by taking $Z = X_{\text{red}}$. In the event X is reducible and $X = Z_1 \cup Z_2$ where Z_i are proper closed, since the conclusion of the theorem also holds for $X - Z_1 = Z_2 - (Z_1 \cap Z_2)$ as it holds for Z_2 and $Z_1 \cap Z_2$ which are proper closed subschemes of X, it therefore will hold for X by virtue of the localization exact sequence for the triple $(X, Z_1, X - Z_1)$. We are now reduced to the case where X is reduced and irreducible. On account of the functorial property of the location sequences, we take the direct limit of the above diagram as Z runs over all proper closed subschemes of X ordered by inclusion. Since direct limits are exact and $\varinjlim CK_{p,q}(U) = CK_{p-d,q+d}(k(X))$, we have the commutative diagram

$$\begin{array}{ccccc} \cdots \varinjlim CK_{p,q}(Z) & \longrightarrow & CK_{p,q}(X) & \longrightarrow & CK_{p-d,q+d}(k(X)) \cdots \\ \cong\downarrow & & \downarrow & & \downarrow \\ \cdots \varinjlim CK_{p+n,q-n}(V_Z) & \longrightarrow & CK_{p+n,q-n}(V) & \longrightarrow & CK_{p-d+n,q+d-n}(V_{k(X)}) \cdots \end{array}$$

where both rows are exact by virtue of which the proof is reduced to the case that $X = \text{spec}(k)$ where k is a field and V an affine space $\mathbb{A}^n$ over k. We have already ascertained that $\pi^* : CK_{p,q}(k) \cong CK_{p+1,q-1}(\mathbb{A}^1)$ from which it follows that $CK_{p,q}(X) \cong CK_{p+1,q-1}(X \times \mathbb{A}^1)$ for all $X \in \mathbf{QP}_k$. Therefore $CK_{p,q}(k) \cong CK_{p+n,q-n}(\mathbb{A}^n)$ and the proof of the theorem is complete. ■

6. Some properties of CK-homology

6.1. $CK(X)$ is a module over $K^V(X)$

Given $\mathcal{A}, \mathcal{B}, \mathcal{C}$ exact categories and $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ a bi-exact functor, F.Waldhausen has defined a product structure [10] $K_i(\mathcal{A}) \otimes K_j(\mathcal{B}) \to K_{i+j}(\mathcal{C})$ for all $i, j \in \mathbb{Z}$. In fact, this product structure on the K-groups is associative and natural with respect to pull-backs and push-forwards and localization sequences of K-groups.

Let $\mathcal{P}(X)$ be the exact category of finitely generated locally free coherent $\mathcal{O}_X$-modules and denote by $K_i^V(X)$ the associated Quillen K-groups, the pairing $\mathcal{P}(X) \times \mathcal{M}_p(X) \to \mathcal{M}_p(X)$ given by tensor product over $\mathcal{O}_X$ is bi-exact. This induces a product $K_r^V(X) \otimes K_i(\mathcal{M}_p(X)) \to K_{i+r}(\mathcal{M}_p(X))$. This product commutes with maps in (1) and descends to

$$K_r^V(X) \otimes CK_{p,q}(X) \to CK_{p,q+r}(X).$$

Theorem 6.1 (*The Projection Formula*). *Let f be a proper morphism between two quasi-projective schemes X and Y, $a \in K_r^V(Y)$ and $x \in CK_{p,q}(X)$, we have the formula*

$$f_*(f^*(a) \cdot x) = a \cdot f_*(x).$$

Proof. Let $\mathcal{E} \in \mathcal{P}(X)$, $\mathcal{F} \in \mathcal{M}_p(X)$. Compare two maps

$$\begin{array}{ccc} \mathcal{P}(Y) \times F_p(X, f) & \longrightarrow & F_p(X, f) \\ & & \downarrow{\scriptstyle f_*} \\ \mathcal{P}(Y) \times F_p(X, f) & \longrightarrow & \mathcal{M}_p(Y) \end{array}$$

where the top row is given by $f^*(\mathcal{E}) \times \mathcal{F} \to f^*(\mathcal{E}) \otimes \mathcal{F}$ followed by the vertical arrow f_* while the bottom row is given by $\mathcal{E} \times f_*(\mathcal{F}) \to \mathcal{E} \otimes f_*(\mathcal{F})$. Note since $f^*(\mathcal{E})$ is locally free on X, we have $f^*(\mathcal{E}) \otimes \mathcal{F} \in F_p(X, f)$. The two maps are equal by the projection formula for sheaves $f_*(f^*(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F}) \cong \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$. The pairings on both rows are bi-exact and the column is exact which give rise to two equal maps $K_r^V(Y) \otimes K_i(\mathcal{M}_p(X)) \to K_{i+r}(\mathcal{M}_p(X)) \to K_{i+r}(\mathcal{M}_p(Y))$ where the second arrow is f_* and $K_r^V(Y) \otimes K_i(\mathcal{M}_p(X)) \to K_{i+r}(\mathcal{M}_p(Y))$. The equality of these maps descend to CK-groups: $K_r^V(Y) \otimes CK_{p,q}(X) \to CK_{p,q+r}(X) \to CK_{p,q+r}(Y)$ and $K_r^V(Y) \otimes CK_{p,q}(X) \to CK_{p,q+r}(Y)$. On the level of elements, this is the projection formula in question. ■

6.2. $CK(X)$ as a module over $CK(k)$

Let X, Y be two quasi-projective schemes and $\mathcal{F} \in \mathcal{M}_p(X), \mathcal{G} \in \mathcal{M}_q(Y)$. $\mathcal{F} \times \mathcal{G}$ is a sheaf on $X \times Y$ whose support has dimension less than $p+q$. This means we have a paring $\mathcal{M}_p(X) \times \mathcal{M}_q(Y) \to \mathcal{M}_{p+q}(X \times Y)$ which is bi-exact, hence there are [10] for all $i, j \in \mathbb{Z}$, $K_i(\mathcal{M}_p(X)) \otimes K_j(\mathcal{M}_q(Y)) \to K_{i+j}(\mathcal{M}_{p+q}(X \times Y))$. By the functoriality of products between K-groups, the following diagram is commutative

$$\begin{array}{ccc} K_i(\mathcal{M}_p(X)) \otimes K_j(\mathcal{M}_q(Y)) & \longrightarrow & K_{i+j}(\mathcal{M}_{p+q}(X \times Y)) \\ \downarrow & & \downarrow \\ K_i(\mathcal{M}_{p+1}(X)) \otimes K_j(\mathcal{M}_q(Y)) & \longrightarrow & K_{i+j}(\mathcal{M}_{p+q+1}(X \times Y)) \end{array}$$

which induces $CK_{p,p'}(X) \otimes D^1_{q,q'}(Y) \to CK_{p+q,p'+q'}(X \times Y)$ where $p' = i - p$, $q' = j - q$. Similarly, there is $D^1_{p,p'}(X) \otimes CK_{q,q'}(Y) \to CK_{p+q,p'+q'}(X \times Y)$. As $CK_{p,p'}(X), CK_{q,q'}(Y)$ are quotients of $D^1_{p,p'}(X)$ and $D^1_{q,q'}(Y)$ respectively and both products have as their origin $D^1_{p,p'}(X) \otimes D^1_{q,q'}(Y) \to CK_{p+q,p'+q'}(X \times Y)$, one concludes that they descend to a product structure on the quotients

$$CK_{p,p'}(X) \otimes CK_{q,q'}(Y) \to CK_{p+p',q+q'}(X \times Y). \tag{5}$$

Owing to the nature of the commutation rule of K-group products in general, for $a \in CK_{p,p'}(X)$ and $b \in CK_{q,q'}(Y)$,

$$a \times b = (-1)^{(p+p')(q+q')} b \times a. \tag{6}$$

By taking $X = Y = \mathrm{spec}(k)$, one sees that $CK(k)$ forms a ring. Let $Y = \mathrm{spec}(k)$, one acquires a module structure on $CK(X)$ over $CK(k)$: $CK_{p,p'}(k) \otimes CK_{q,q'}(X) \to CK_{p+p',q+q'}(X)$.

6.3. Gysin homomorphisms

Let $i : Y \hookrightarrow X$ be a regular closed embedding of codimension d. In this section we aim to define the pull-back $i^* : CK_{p,q}(X) \to CK_{p-d,q+d}(Y)$.

Denote the ideal sheaf of Y by $\mathcal{I}$. There is the construction of deformation to the normal cone associated with i [2]: In the event that $X = \mathrm{spec}(A)$ and Y is given by the ideal I. Consider $\widetilde{A} = \cdots \oplus I^2 t^{-2} \oplus I t^{-1} \oplus A \oplus At \oplus \cdots$, a subring of $A[t, t^{-1}]$. We have $\widetilde{A}/(t) = \cdots \oplus (I^2/I^3)t^{-2} \oplus (I/I^2)t^{-1} \oplus A/I$ and $\widetilde{A}_t = A[t, t^{-1}]$. *The deformation variety D_i* is defined to be $\mathrm{spec}(\widetilde{A})$. D_i admits a canonical morphism to $\mathbb{A}^1_k = \mathrm{spec}(k[t])$ whose fibre over $t = 0$ is *the deformation cone C_i* $= \mathrm{spec}(\widetilde{A}/(t))$. The open complement of C_i in D_i is $X \times \mathbb{G}_m = \mathrm{spec}(\widetilde{A}_t)$, where $\mathbb{G}_m = \mathrm{spec}(k[t, t^{-1}])$. In other words, the fibres of D_i over $\mathbb{A}_1$ are X at points apart from 0. We denote by p the natural projection $X \times \mathbb{G}_m \to X$. Observe that there is also a canonical morphism $\mu : C_i \to X$ which factors through $\pi : C_i \to Y$.

The pull-back i^* will be obtained by a sequence of maps. The localization sequence associated to the triple $C_i \hookrightarrow D_i \hookleftarrow X \times \mathbb{G}_m$ provides us with a connecting homomorphism $\delta : CK_{p+1,q}(X \times \mathbb{G}_m) \to CK_{p,q}(C_i)$. There is a distinguished invertible element t in $K_1^V(X \times \mathbb{G}_m)$ which is given by the image of the element $t \in K_1^V(\mathbb{G}_m) = k[t, t^{-1}]^\times$ under the pull-back $K_1^V(\mathbb{G}_m) \to K_1^V(X \times \mathbb{G}_m)$. $CK(X \times \mathbb{G}_m)$ is a module over $K^V(X \times \mathbb{G}_m)$ and multiplication by t gives us an invertible morphism $CK_{p,q-1}(X \times \mathbb{G}_m) \to CK_{p,q}(X \times \mathbb{G}_m)$. Assembling these maps we arrive at [7]

$$\begin{tikzcd} CK_{p,q}(X) \arrow[d, "p^*"'] & CK_{p,q}(C_i) \\ CK_{p+1,q-1}(X \times \mathbb{G}_m) \arrow[r, "\times t"] & CK_{p+1,q}(X \times \mathbb{G}_m) \arrow[u, "\delta"'] \end{tikzcd}$$

the composition of which will be labeled by $\sigma_i : CK_{p,q}(X) \to CK_{p,q}(C_i)$.

When i is a regular embedding of codimension d, I/I^2 is a locally free A/I-module of rank d and $I^n/I^{n+1} \cong S^n(I/I^2)$. In this case $C_i = \text{spec}(\oplus_{n\geqslant 0} I^n/I^{n+1}) = \text{spec}(S^{\cdot}(I/I^2))$ which is recognized as the normal bundle $N_Y X$ of Y in X and $\pi : C_i = N_Y X \to Y$ is the canonical projection. The homotopy invariance property now dictates that $\pi^* : CK_{p-d,q+d}(Y) \to CK_{p,q}(N_Y X)$ is an isomorphism and we obtain $(\pi^*)^{-1} : CK_{p,q}(C_i) \to CK_{p-d,q+d}(Y)$. We define, for an regular embedding i of codimension d,

$$i^* = (\pi^*)^{-1} \circ \sigma_i : CK_{p,q}(X) \to CK_{p-d,q+d}(Y).$$

This is the *Gysin homomorphism*.

Remark 6.2. The Gysin morphism satisfies various functorial properties. Let

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

be a fibre diagram with i and i' regular embeddings of codimension d.

1. (Push-forward) If f is proper, then $i^* f_* = f'_* i'^* : CK_{p,q}(X') \to CK_{p-d,q+d}(Y)$.
2. (Pull-back) If f is flat of codimension r, then $i'^* f^* = f'^* i^* : CK_{p,q}(X) \to CK_{p+r-d,q-r+d}(Y')$.

These directly follow from the definition of the Gysin morphism and the functorial properties of the f_* and f^*, viz., they commute with all maps constituting the composite of the Gysin morphism.

3. (composition rule) Let $i_1 : Z \hookrightarrow Y$ and $i_2 : Y \hookrightarrow X$ be two successive regular embeddings of codimension d_1 and d_2 respectively, then $i_1^* i_2^* = (i_2 i_1)^*$.

To see this, compare the Gysin morphisms on both sides. The normal cones to the embeddings $i : N_{i_2}|_Z \hookrightarrow N_{i_2}$ and $j : N_{i_1} \hookrightarrow N_{i_2 i_1}$ are canonically isomorphic which we shall denote by N. The assertion in question ensues from the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma_{i_2 i_1}"] \arrow[d, "\sigma_{i_2}"'] & N_{i_2 i_1} \arrow[d, "\sigma_j"'] & \\ N_{i_2} \arrow[r, "\sigma_i"] & N & Z \arrow[l, "(p_{i_1} p_j)^*"] \arrow[ul, "p^*_{i_2 i_1}"'] \arrow[dl, "p^*_{i_1}"] \\ Y \arrow[u, "p^*_{i_2}"] \arrow[r, "\sigma_{i_1}"] & N_{i_i} \arrow[u, "p^*_j"] & \end{tikzcd}$$

That the lower square and triangle commute is a direct consequence of functorial properties of pull-backs. It is straightforward to verify that for a closed embedding $Z \hookrightarrow X$ and a flat morphism $h : X \to Y$, assuming that the composition $g : C_i \to Z \hookrightarrow X \to Y$ is flat, then $g^* = \sigma_i h^*$. Hence the upper triangle commutes. Finally, let us sketch a reason for the commutativity of the top square. There is a double deformation scheme D associated to i_1 and

i_2 (c.f. [7]) with the following commutative diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & D_i \arrow[d] & N_{i_2} \times \mathbb{G}_m \arrow[l] \arrow[d] \\
D_j \arrow[r] & D & D_{i_2} \times \mathbb{G}_m \arrow[l] \\
\mathbb{G}_m \times N_{i_2 i_1} \arrow[r] \arrow[u] & \mathbb{G}_m \times D_{i_2 i_1} \arrow[u] & \mathbb{G}_m \times X \times \mathbb{G}_m \arrow[l] \arrow[u]
\end{tikzcd}$$

Each row and column gives rise to a homotopy fibration of the corresponding classifying spaces for K-groups and it is a fact of connecting morphisms for homotopy groups that the connecting morphisms along the edges of the square obey $\partial_t \partial_r = -\partial_l \partial_b$ where the subscript t stands for "top" etc. Now $\sigma_i \sigma_{i_2} = \partial_t \circ \{\times t\} \circ p^*_{N_{i_2}} \circ \sigma_{i_2} = \partial_t \circ \{\times t\} \circ \sigma_{i_2 \times \mathbb{G}_m} \circ p^*_X = \partial_t \circ \{\times t\} \circ \partial_r \circ \{\times t'\} \circ p^* = -\partial_t \circ \partial_r \circ \{\times t\} \circ \{\times t'\} \circ p^*$ where $p : \mathbb{G}_m \times X \times \mathbb{G}_m \to X$ is the projection and t, t' are pull-backs of the distinguished element in the two copies of $K_1^V(\mathbb{G}_m)$ respectively. The last equality is true on account of the graded commutativity between boundary maps and the product on K-groups. Similarly, $\sigma_j \sigma_{i_2 i_1} = -\partial_l \circ \partial_b \circ \{\times t'\} \circ \{\times t\} \circ p^*$. Note that $\{\times t'\} \circ \{\times t\} = -\{\times t\} \circ \{\times t'\}$ since both t, t' are elements in $K_1^V(\mathbb{G}_m \times X \times \mathbb{G}_m)$. Hence the commutativity in question is attained.

6.4. $CK(X)$ is a ring for smooth X

Let X be a smooth quasi-projective scheme over k of dimension d. The diagonal of X: $\Delta : X \hookrightarrow X \times X$ is regular and we have the Gysin morphism $\Delta^* : CK_{p,q}(X \times X) \to CK_{p-d,q+d}(X)$. Taking $Y = X$ in (5) and composing it with Δ^*, we obtain a product on $CK(X)$: $CK_{p,p'}(X) \otimes CK_{q,q'}(X) \to CK_{p+p',q+q'}(X \times X) \to CK_{p+p'-d,q+q'+d}(X)$.

Introduce the cohomology notation $CK^{p,q}(X) = CK_{d-p,-d-q}(X)$. The above pairing reads $CK^{p,p'}(X) \otimes CK^{q,q'}(X) \to CK^{p+p',q+q'}(X)$, which by definition is evidently associative. On account of the product rule (6) this pairing turns $CK'(X) = \coprod_{p,q \in \mathbb{Z}} CK^{p,q}(X)$ into a bi-graded commutative ring and the $CK(X)$ a bi-graded module over it. The module structure is given by $CK^{i,i'}(X) \cdot CK_{p,p'}(X) \to CK_{p-i,p'-i'}(X)$. The identity element $1 = [\mathcal{O}_X]$ lies in $CK^{0,0}(X) = CK_{d,-d}(X)$.

In the event $f : X \to Y$ is a flat morphism between quasi-projective schemes of fixed dimension, we have $f^* : CK^{p,q}(Y) \to CK^{p,q}(X)$. By 2 of Remark 6.2, f^* induces a homomorphism $CK'(Y) \to CK'(X)$ between bi-graded commutative rings.

6.5. The projective bundle theorem

Let L be a line bundle on X, By definition $c_1(\mathcal{L}) = (\pi^*)^{-1} \circ s_*$ where $\pi : L \to X$ is the canonical projection and s is a section of L.

Lemma 6.1. *Let $i : Z \hookrightarrow X$ be a closed embedding with $j : U \hookrightarrow X$ the open complement and δ is the connecting homomorphism $CK_{p,q}(U) \to CK_{p-1,q+1}(Z)$ in the localization exact sequence. Then $c_1(\mathcal{L}|_Z) \circ \delta = \delta \circ c_1(\mathcal{L}|_U)$.*

Proof of the Lemma. We show both constituents of $c_1(\mathcal{L})$ commutes with connecting morphisms in some proper sense. Let δ^L be the corresponding connecting homomorphism for the triple $(L, L|_Z, L|_U)$. By Theorem 5.2, $\pi |_Z^* \circ \delta = \delta^L \circ \pi |_U^*$, hence $\delta \circ (\pi |_U^*)^{-1} = (\pi |_Z^*)^{-1} \circ \delta^L$. By examining

$$\begin{tikzcd}
L|_Z \arrow[r] & L & L|_U \arrow[l] \\
Z \arrow[r] \arrow[u, "s|_Z"] & X \arrow[u, "s"] & U \arrow[l] \arrow[u, "s|_U"]
\end{tikzcd}$$

and following a similar argument to the one in the proof of Theorem 5.2, one concludes $s|_{Z*} \circ \delta = \delta^L \circ s|_{U*}$. □

The next proposition gives an explicit description of the $c_1(\mathcal{L})$ for K-groups.

Proposition 6.1. *$c_1(\mathcal{L}) = (1-\mathcal{L}^\vee)\beta^{-1} : K_{p,q}(X) \to K_{p-1,q+1}(X)$ where the right-hand side is the morphism given by multiplication.*

Proof of the Proposition. Let us compute $s_*(\mathcal{F})$ for $\mathcal{F} \in \mathcal{M}(X)$. In $\mathcal{M}(L)$, there is the short exact sequence

$$0 \to \pi^*\mathcal{L}^\vee \to \mathcal{O}_L \to s_*(\mathcal{O}_X) \to 0.$$

In case $X = \mathrm{spec}(A)$ with A a local ring, $L = \mathrm{spec}(A[t])$ where $A[t]$ is the polynomial ring over A and $\mathcal{F}$ is given by a finite A-module M. The above short sequence is

$$0 \longrightarrow A[t] \xrightarrow{\times t} A[t] \longrightarrow A \longrightarrow 0$$

which splits. Tensor the short exact sequence with $\pi^*\mathcal{F}$, we get

$$0 \to \pi^*\mathcal{F} \otimes_{\mathcal{O}_L} \pi^*\mathcal{L}^\vee \to \pi^*\mathcal{F} \to s_*\mathcal{O}_X \otimes_{\mathcal{O}_L} \pi^*\mathcal{F} \to 0$$

which is also exact since the original sequence splits locally. The term on the left, $\pi^*\mathcal{F} \otimes_{\mathcal{O}_L} \pi^*\mathcal{L}^\vee \cong \mathcal{O}_L \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_L} \mathcal{O}_L \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_L \otimes_{\mathcal{O}_X} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee) \cong \pi^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee)$. Observe that the natural morphism $s_*\mathcal{F} \to s_*\mathcal{O}_X \otimes_{\mathcal{O}_L} \pi^*\mathcal{F}$ is an isomorphism since locally $s_*\mathcal{O}_X \otimes_{\mathcal{O}_L} \pi^*\mathcal{F}$ is $A \otimes_{A[t]} A[t] \otimes_A M \cong M$, a $A[t]$ module via trivial t action, coinciding with $s_*\mathcal{F}$. In other words, the above short exact sequence can be rewritten as $0 \to \pi^*(\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}) \to \pi^*\mathcal{F} \to s_*\mathcal{F} \to 0$. Hence we obtain a short exact sequence of functors from $\mathcal{M}(X)$ to $\mathcal{M}(L)$: $0 \to \pi^*\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \to \pi^* \to s_* \to 0$ which induces [6] an equality of morphisms between groups $s_* = \pi^* \circ (1 - \mathcal{L}^\vee) : K_i(X) \to K_i(L)$. Since $\mathcal{L}^\vee$ represents an element in $K_0(X)$, the map $\mathcal{L}^\vee$ is given by the product. It follows from this that $(\pi^*)^{-1} \circ s_* = (1 - \mathcal{L}^\vee) : K_i(X) \to K_i(X)$. On account of the fact that $K_{p,q}(X) = K_{p+q}(X)\beta^p$, one concludes with $c_1(\mathcal{L}) = (1 - \mathcal{L}^\vee)\beta^{-1} : K_{p,q}(X) \to K_{p-1,q+1}(X)$. $\square$

Lemma 6.2. *Let s be a global section of L which is nontrivial on any open subset of X. Let Z be the vanishing of s. Given a commutative diagram*

$$\begin{tikzcd} Z \arrow[rr, "i"] \arrow[dr, "p_Z"'] & & X \arrow[dl, "p_X"] \\ & Y & \end{tikzcd}$$

where p_X, p_Z are flat of fixed codimension. Then $c_1(\mathcal{L})p_X^* = i_*p_Z^*$ on the K-groups.

Proof of the Lemma. From the definitions, $i_*\mathcal{O}_Z$ on X admits the following resolution $0 \to \mathcal{L}^\vee \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$ where the second arrow is given by $\otimes s$. All terms here are flat $\mathcal{O}_Y$-modules and there is in turn a short exact sequence of functors from $\mathcal{M}(Y)$ to $\mathcal{M}(X)$: $0 \to \mathcal{L}^\vee \otimes_{\mathcal{O}_X} p_X^* \to p_X^* \to i_*\mathcal{O}_Z \otimes_{\mathcal{O}_X} p_X^* \to 0$. Observe that $i_*\mathcal{O}_Z \otimes_{\mathcal{O}_X} p_X^* = i_*p_Z^*$ and the previous short exact sequence induces $i_*p_Z^* = (1 - \mathcal{L}^\vee)p_X^* : K_i(Y) \to K_i(X)$. Keeping track of the correct indices' shifts, we conclude that $i_*p_Z^* = (1 - \mathcal{L}^\vee)\beta^{-1} \circ p_X^*$ or $i_*p_Z^* = c_1(L)p_X^*$. $\square$

Let $p_n : \mathbb{P}^n \to \mathrm{spec}(k)$ be the structure morphism and $i : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ a closed embedding and $j : \mathbb{A}^n \to \mathbb{P}^n$ the open complement. The homotopy invariance of K-groups says that $k_{p,q}(k) \to K_{p+n,q-n}(\mathbb{A}^n)$ is an isomorphism for all $n \in \mathbb{N}$.

Proposition 6.2. *There is a homomorphism $\tau : K_{p,q}(\mathbb{A}^n) \to K_{p,q}(\mathbb{P}^n)$ which splits the localization exact sequence of K-groups $\cdots \to K_{p,q}(\mathbb{P}^{n-1}) \to K_{p,q}(\mathbb{P}^n) \to K_{p,q}(\mathbb{A}^n) \to K_{p-1,q}(\mathbb{P}^{n-1}) \to \cdots$. Consequently, we have an isomorphism $\bigoplus_{m=0}^n K_{p+m-n,q-m+n}(k) \to K_{p,q}(\mathbb{P}^n)$ which, on the mth component, is given by $c_1(\mathcal{O}_{\mathbb{P}^n}(1))^m \circ p_n^* : K_{p+m-n,q-m+n}(k) \to K_{p,q}(\mathbb{P}^n)$.*

Proof of the Proposition. Start with the commutative diagram

$$\begin{tikzcd} \mathbb{A}^n \arrow[rr, "j"] \arrow[dr, "\pi_n"'] & & \mathbb{P}^n \arrow[dl, "p_n"] \\ & k & \end{tikzcd}$$

and define $\tau = (p_n^* \circ (\pi_n^*)^{-1}) : K_{p,q}(\mathbb{A}^n) \to K_{p,q}(\mathbb{P}^n)$. As $j^* \circ p_n^* = \pi_n^*$, $j^* \circ \tau$ is the identity map and the long exact sequence splits.

To obtain the announced recipe of the isomorphism, we induct on n for a fixed pair (p,q). The case $n = 0$ is trivial.

We examine the induction step $n-1$ to n. The splitting of the localization sequence gives rise to an isomorphism $(i_*, \tau \circ \pi_n^*) : K_{p,q}(\mathbb{P}^{n-1}) \oplus K_{p-n,q+n}(k) \to K_{p,q}(\mathbb{P}^n)$. Since $p_n^* = \tau \circ \pi_n^*$, we simply verified the conclusion on the 0th component. The induction hypothesis asserts that $\bigoplus_{m=1}^n K_{p+m-n,q-m+n}(k) \to K_{p,q}(\mathbb{P}^{n-1})$ is an isomorphism which, on the mth component is given by $c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{m-1} \circ p_{n-1}^*$. We need to compose this isomorphism with i_*. By the push-forward property of Chern classes and that $i^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, $i_* \circ c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{m-1} \circ p_{n-1}^* = c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{m-1} \circ i_* \circ p_{n-1}^*$. By Lemma 6.2, $i_* \circ p_{n-1}^* = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \circ p_n^*$. We conclude that on the mth component, the isomorphism is given by $c_1(\mathcal{O}_{\mathbb{P}^n}(1))^m \circ p_n^*$. $\square$

Theorem 6.3 (*Projective Bundle Theorem*). *Let X be a quasi-projective scheme over k, E a vector bundle of rank $r+1$ on X and $p : \mathbb{P}(E) \to X$ the associated projective bundle. Let $\mathcal{O}(1)$ be the canonical bundle on $\mathbb{P}(E)$. We define, for $m \geqslant 0$, $\alpha_m = (-c_1(\mathcal{O}(-1)))^m \circ p^* : CK_{p-r+m,q+r-m}(X) \to CK_{p+m,q-m}(\mathbb{P}(E)) \to CK_{p,q}(\mathbb{P}(E))$. Then $\oplus_{i=0}^r \alpha_m : \oplus_{m=0}^r CK_{p-r+m,q+r-m}(X) \to CK_{p,q}(\mathbb{P}(E))$ is an isomorphism.*

Proof. For conciseness, we denote $\xi = -c_1(\mathcal{O}(-1))$ and $\tau = c_1(\mathcal{O}(1))$. For a triple $Z \hookrightarrow X \hookleftarrow U$ as before,

$$\begin{tikzcd}
\mathbb{P}_Z(E) \arrow[r, "i^P"] \arrow[d, "p_Z"] & \mathbb{P}_X(E) \arrow[d, "p_X"] & \mathbb{P}_U(E) \arrow[l, "j^P"'] \arrow[d, "p_U"] \\
Z \arrow[r, "i"] & X & U \arrow[l, "j"']
\end{tikzcd}$$

gives rise to a commutative diagram of localization sequences

$$\begin{tikzcd}
{} \arrow[r, "i_*"] & CK_{p-r+m,q+r-m}(X) \arrow[r, "j^*"] \arrow[d, "p^*"'] & CK_{p-r+m,q+r-m}(U) \arrow[r, "\delta"] \arrow[d, "p_U^*"'] & CK_{p-r+m-1,q+r-m}(Z) \arrow[r] \arrow[d, "p_Z^*"'] & {} \\
{} \arrow[r, "i_*^P"] & CK_{p+m,q-m}(\mathbb{P}(E)) \arrow[r, "j^{P*}"] \arrow[d, "\xi_X^m"'] & CK_{p+m,q-m}(\mathbb{P}_U(E)) \arrow[r, "\delta^P"] \arrow[d, "\xi_U^m"'] & CK_{p+m-1,q-m}(\mathbb{P}_Z(E)) \arrow[r] \arrow[d, "\xi_Z^m"'] & {} \\
{} \arrow[r, "i_*^P"] & CK_{p,q}(\mathbb{P}(E)) \arrow[r, "j^{P*}"] & CK_{p,q}(\mathbb{P}_U(E)) \arrow[r, "\delta^P"] & CK_{p-1,q}(\mathbb{P}_Z(E)) \arrow[r] & {}
\end{tikzcd}$$

for all nonnegative integers m. Squares in the lower row commute by functorial properties of Chern classes, e.g., the lower right square commutes by Lemma 6.1. As before, by inducting on the dim X and taking the direct limit of all such diagrams with respect to closed subschemes of X, the problem is reduced to the case of $X = \mathrm{spec}(k)$.

We now show $\oplus_{m=0}^r CK_{p-r+m,q+r-m}(k) \to CK_{p,q}(\mathbb{P}^r)$ is an isomorphism by inducting on r. Case $r = 1$. On account of natural morphisms $CK_{p,q} \to K_{p,q} \cong K_{p+q}$, there is the following commutative diagram of localization sequences

$$\begin{tikzcd}
\cdots \arrow[r] & CK_{p,q}(k) \arrow[r] \arrow[d] & CK_{p,q}(\mathbb{P}^1) \arrow[r] \arrow[d] & CK_{p,q}(\mathbb{A}^1) \arrow[r] \arrow[d] & CK_{p-1,q}(k) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{p,q}(k) \arrow[r] & K_{p,q}(\mathbb{P}^1) \arrow[r] & K_{p,q}(\mathbb{A}^1) \arrow[r] & K_{p-1,q}(k) \arrow[r] & \cdots .
\end{tikzcd}$$

By homotopy invariance, $CK_{p,q}(\mathbb{A}^1) \cong CK_{p-1,q+1}(k)$, $K_{p,q}(\mathbb{A}^1) \cong K_{p-1,q+1}(k)$ and the bottom sequence splits: $K_{p,q}(\mathbb{P}^1) \cong K_{p,q}(k) \oplus K_{p-1,q+1}(k)$. Therefore the above diagram can be rewritten as

$$\begin{tikzcd}
\cdots \arrow[r] & CK_{p+1,q}(k) \arrow[r, "0"] \arrow[d] & CK_{p,q}(k) \arrow[r] \arrow[d] & CK_{p,q}(\mathbb{P}^1) \arrow[r] \arrow[d] & CK_{p-1,q+1}(k) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{p+1,q}(k) \arrow[r, "0"] & K_{p,q}(k) \arrow[r] & K_{p,q}(\mathbb{P}^1) \arrow[r] & K_{p,q}(k) \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 5.1 says $CK_{p,q}(k) \cong K_{p,q}(k)$ for $p \geqslant 0$ and 0 otherwise. In any event $CK_{p,q}(\mathbb{P}^1) \cong CK_{p,q}(k) \oplus CK_{p-1,q+1}(k)$. For the induction step $r \to r+1$, consider a similar commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & CK_{p+1,q}(\mathbb{A}^{r+1}) \arrow[r] \arrow[d] & CK_{p,q}(\mathbb{P}^r) \arrow[r] \arrow[d] & CK_{p,q}(\mathbb{P}^{r+1}) \arrow[r] \arrow[d] & CK_{p,q}(\mathbb{A}^{r+1}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & K_{p+1,q}(\mathbb{A}^{r+1}) \arrow[r] & K_{p,q}(\mathbb{P}^r) \arrow[r] & K_{p,q}(\mathbb{P}^{r+1}) \arrow[r] & K_{p,q}(\mathbb{A}^{r+1}) \arrow[r] & \cdots . \end{tikzcd}$$

By the induction hypothesis, $CK_{p,q}(\mathbb{P}^r) \cong \oplus_{m=0}^r CK_{p-r+m,q+r-m}(k)$; by homotopy invariance, $CK_{p,q}(\mathbb{A}^{r+1}) \cong CK_{p-r-1,q+r+1}(k)$. It is also established that $K_{p,q}(\mathbb{P}^r) \cong \oplus_{m=0}^r K_{p-r+m,q+r-m}(k)$ and $K_{p,q}(\mathbb{A}^{r+1}) \cong K_{p-r-1,q+r+1}(k)$. By Proposition 6.2 the bottom long exact sequence splits $K_{p,q}(\mathbb{P}^{r+1}) \cong K_{p,q}(\mathbb{P}^r) \oplus K_{p,q}(\mathbb{A}^{r+1})$, and the above diagram is isomorphic to

$$\begin{tikzcd} \arrow[r] & CK_{p-r,q+r+1}(k) \arrow[r, "0"] \arrow[d] & \coprod_{m=0}^r CK_{p-r+m,q+r-m}(k) \arrow[r] \arrow[d] & CK_{p,q}(\mathbb{P}^{r+1}) \arrow[r] \arrow[d] & CK_{p-r-1,q+r+1}(k) \arrow[r] \arrow[d] & {} \\ \arrow[r] & K_{p-r,q+r+1}(k) \arrow[r, "0"] & \coprod_{m=0}^r K_{p-r+m,q+r-m}(k) \arrow[r] & K_{p,q}(\mathbb{P}^{r+1}) \arrow[r] & K_{p-r-1,q+r+1}(k) \arrow[r] & {} \end{tikzcd}$$

Proposition 5.1 implies $CK_{p,q}(k) \hookrightarrow K_{p,q}(k)$ and all the vertical arrows are injections except for the third one, and so it is also so. Therefore the top exact sequence also splits and we have $CK_{p,q}(\mathbb{P}^{r+1}) \cong \oplus_{m=0}^{r+1} CK_{p-r-1+m,q+r+1-m}(k)$.

It remains to show that this isomorphism is indeed $\oplus\alpha_m$. From the proof of Proposition 6.2, the corresponding isomorphism for K-groups $\oplus_{m=0}^r K_{p-r+m,q+r-m}(k) \to K_{p,q}(\mathbb{P}^r)$ resulting from the bottom long exact sequence is actually given, on each component, by

$$K_{p-r+m,q+r-m}(k) \xrightarrow{p^*} K_{p+m,q-m}(\mathbb{P}^r) \xrightarrow{\tau_K^i} K_{p,q}(\mathbb{P}^r).$$

Owing to the functorial properties of push-forwards and pull-backs and the definition of Chern classes, the following diagram is commutative

$$\begin{tikzcd} CK_{p-r+m,q+r-m}(k) \arrow[r, "p^*"] \arrow[d] & CK_{p+m,q-m}(\mathbb{P}^r) \arrow[r, "\tau_{CK}^m"] \arrow[d] & CK_{p,q}(\mathbb{P}^r) \arrow[d] \\ K_{p-r+m,q+r-m}(k) \arrow[r, "p^*"] & K_{p+m,q-m}(\mathbb{P}^r) \arrow[r, "\tau_K^m"] & K_{p,q}(\mathbb{P}^r). \end{tikzcd}$$

Identifying $CK_{p,q}(\mathbb{P}^r)$ as a subgroup of $K_{p,q}(\mathbb{P}^r)$, we conclude the through map on the top row coincides with the corresponding arrow in the isomorphism $\oplus_{m=0}^r CK_{p-r+m,q+r-m}(k) \to CK_{p,q}(\mathbb{P}^r)$.

We demonstrate that τ can be replaced by $-\xi$ in the isomorphism. Denote $\gamma_m = \tau^m \circ p^*$. From the last paragraph, it suffices to show the equality for K-theory. The group law for K-theory assumes $F(u,v) = u + v - \beta uv$ and the inverse of u is given by $u^{-1} = -u(1 + \beta u + \beta^2 u^2 + \cdots)$ (c.f. Section 6.6). Therefore, $\xi = -\tau(1 + \beta\tau + \beta^2\tau^2 + \cdots)$ and $\psi = 1 + \beta\tau + \beta^2\tau^2 + \cdots$ is an invertible or auto-morphism of $K_{p,q}(\mathbb{P}^r)$. Now $\alpha_m = \psi^m \circ \gamma_m$ and we need to show $\oplus_{m=0}^r \alpha_m$ is an isomorphism. $\oplus_{m=0}^r \gamma_m$ being an isomorphism, there are $p_m : K_{p,q}(\mathbb{P}^r) \to K_{p-r+m,q+r-m}(k)$ such that $p_n \circ \gamma_m = \delta_{mn}$ and $\sum_{m=0}^r \gamma_m \circ p_m = 1_{\mathbb{P}^r}$. Denote $p = \oplus_{m=0}^r p_m$, the inverse of $\oplus_{m=0}^r \gamma_m$. It suffices to show $p \circ \left(\oplus_{m=0}^r \alpha_m\right)$ is an auto-morphism on $\oplus_{m=0}^r K_{p-r+m,q+r-m}(k)$. Since $\beta\tau$ is a nilpotent endomorphism on $K_{p,q}(\mathbb{P}^r)$, $p_n \circ \psi^m \circ \gamma_m = \delta_{mn} + p_n \circ N_m \circ \gamma_m$ where N_m is a power series in $\beta\tau$ without constant, hence nilpotent on $K_{p,q}(\mathbb{P}^r)$. Therefore $p \circ \left(\oplus_{m=0}^r \alpha_m\right) = \oplus_{m,n=0}^r (p_n \circ \psi^m \circ \gamma_m) = 1 + \oplus_{m=0}^r p \circ N_m \circ \gamma_m$ and the second summand is a nilpotent endomorphism on $\oplus_{m=0}^r K_{p-r+m,q+r-m}(k)$ because for $a \in K_{p-r+m,q+r-m}(k)$, $p \circ N_m \circ \gamma_m(a) \in \oplus_{m'<m}^r K_{p-r+m',q+r-m'}(k)$ and the first index of a is lowered at least by 1. It follows that $p \circ \left(\oplus_{m=0}^r \alpha_m\right)$ is invertible. ■

Let us determine the expression of $c_1(\mathcal{O}_{\mathbb{P}^n}(-1))$ under the isomorphism $\oplus_{m=0}^n \alpha_m : \oplus_{m=0}^r CK_{p-n+m,q+n-m}(k) \to CK_{p,q}(\mathbb{P}^n)$. Since $\alpha_i = (-c_1(\mathcal{O}_{\mathbb{P}^n}(-1)))^i \circ p^*$, it is clear that $-c_1(\mathcal{O}_{\mathbb{P}^n}(-1)) \cdot \alpha_i = \alpha_{i+1}$. To make the following diagram commutative

$$\begin{array}{ccc} \oplus_{i=0}^{n} CK_{p-n+i,q+n-i}(k) & \xrightarrow[\sim]{\sum_{i=0}^{r}\alpha_i} & CK_{p,q}(\mathbb{P}^n) \\ {\scriptstyle C_1}\downarrow & & \downarrow{\scriptstyle -c_1(\mathcal{O}_{\mathbb{P}^n}(-1))} \\ \oplus_{i=0}^{n} CK_{p-1-n+i,q+1+n-i}(k) & \xrightarrow[\sim]{\sum_{i=0}^{n}\alpha_i} & CK_{p-1,q+1}(\mathbb{P}^n) \end{array}$$

it is assumed that

$$C_1 = \begin{cases} Id & \text{for } 0 \leqslant i \leqslant n-1 \\ 0 & \text{for } i = n. \end{cases}$$

6.6. *The formal group law for CK-homology*

Let $\pi : X \to \operatorname{spec}(k)$ be quasi-projective of dimension d over k and it defines a class $[X]_{CK} = \pi_*(\mathcal{O}_X) \in CK_{d,-d}(k)$. Observe that $[X]_{CK}$ is the image of $1 = [\mathcal{O}_k]$ under $\pi_* \circ \pi^* : CK_{0,0}(k) \to CK_{d,-d}(k)$. On account of the following commutative diagram in which vertical arrows are isomorphisms

$$\begin{array}{ccccc} CK_{0,0}(k) & \xrightarrow{\pi^*} & CK_{d,-d}(X) & \xrightarrow{\pi_*} & CK_{d,-d}(k) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ K_0(k) & \xrightarrow{\pi^*} & K_0(X) & \xrightarrow{\pi_*} & K_0(k) \end{array}$$

we identify $[X]_{CK}$ with $[X]_K \in K_0(k) \cong \mathbb{Z}$. Recall that $\pi_*([\mathcal{F}]_K) = \sum_{i\geqslant 0}(-1)^i[R^i\pi_*(\mathcal{F})]_K = \sum_{i\geqslant 0}(-1)^i [H^i(\operatorname{spec}(k), \mathcal{F})]_K$. The last equality holds since k is a field. For example, we have $[\mathbb{P}^n]_K = \pi_*(1_{\mathbb{P}^n}) = \sum_{i\geqslant 0}(-1)^i H^i(\operatorname{spec}(k), \mathcal{O}_{\mathbb{P}^n}) = 1_k$ in $K_0(k)$. Compounding this result with the isomorphisms $K_0(k) \to CK_{n,-n}(k)$ which sends 1 to the generator β^n of $CK_{n,-n}(k)$, we conclude that $[\mathbb{P}^n]_{CK} = \beta^n$. In particular, $[\mathbb{P}^1]_{CK} = \beta \in CK_{1,-1}(k)$.

Let $\mathcal{L}$ and $\mathcal{M}$ be two line bundles on X, the formal group law expresses $c_1(\mathcal{L} \otimes \mathcal{M}) = F(c_1(\mathcal{L}), c_1(\mathcal{M}))$. It is shown [5] that for any oriented Borel–Moore homology theory H, F_H can be expressed as a unique power series: $F_H(u, v) = u + v + \sum_{i,j\geqslant 1} a_{ij}u^i v^j \in H_{*,*}(k)[[u, v]]$ with $a_{ij} \in H_{i+j-1,-i-j+1}(k)$. In fact, the a_{ij}s can be determined inductively by classes of $[\mathbb{P}^n]$ and Milnor hypersurfaces $[\mathbb{H}_{n,m}]$.[2] In fact, there are relations (*loc. cit.*):

$$[\mathbb{H}_{n,m}] = [\mathbb{P}^n]\cdot[\mathbb{P}^{m-1}] + [\mathbb{P}^{n-1}]\cdot[\mathbb{P}^m] + \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}[\mathbb{P}^{n-i}]\cdot[\mathbb{P}^{m-j}] \tag{7}$$

in $H(k)$. In particular, since $\mathbb{H}_{1,1} \cong \mathbb{P}^1$, one deduces $a_{11} = -[\mathbb{P}^1]$.

Using the description of Chern classes for K-theory given in Proposition 6.1 and the identity $1 - ab = (1-a) + (1-b) - (1-a)(1-b)$, one quickly checks that the group law for K-theory is: $F_K(u, v) = u + v - \beta uv$. On account of the isomorphism $CK_{n,-n}(k) \to K_{n,-n}(k) \cong K_0(k)$ and Eq. (7), we assert that a_{ij} of CK-homology coincides with that of K-theory. Therefore, we have

Theorem 6.4 (*Formal Group Law for CK-Homology*). $F_{CK}(u, v) = u + v - \beta uv$.

Thus CK-homology is a *multiplicative* oriented Borel–Moore functor [5].

Given a formal group law F_H, there is a unique power series [4] $\chi_H(u) = \sum_{i>0}\alpha_i u^i \in \mathbb{L}[[u]]$, where $\mathbb{L}$ is the Lazard ring, which satisfies the equality $F_H(u, \chi_H(u)) = 0$ and for any line bundle $\mathcal{L}$ on X, $\chi_H(c_1(\mathcal{L})) = c_1(\mathcal{L}^\vee)$. In fact [5], $\chi_H(u) = -u + a_{1,1}u^2 - (a_{1,1})^2u^3 + ((a_{1,1})^3 + a_{1,1}\cdot a_{2,1} + 2a_{3,1} - a_{2,2})u^4 + \cdots = u\cdot\varphi(u)$ and $\varphi(u)$ is a unit in $H(k)[[u]]$.

[2] $[X] \in H_{d,-d}(k)$ with $d = \dim(X)$. Milnor hypersurface $\mathbb{H}_{n,m} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m$ is the smooth closed subscheme defined by the vanishing of a transverse section of the line bundle $p_1^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^m}(1))$ on $\mathbb{P}^n \times \mathbb{P}^m$.

Lemma 6.3. *$F_H(u, \chi_H(v)) = (u - v) \cdot \psi(u, v)$ where $\psi(u, v)$ is a unit in $H(k)[[u, v]]$.*

Proof of the Lemma. Since $F_H(u, \chi_H(u)) = 0$, the kth homogeneous component $F_H(u, \chi_H(u))_k = 0$. Writing $F_H(u, \chi_H(v))_k = \sum_{i+j=k} b_{i,j} u^i v^j$, then $\sum_{i+j=k} b_{i,j} = 0$. Therefore $(u - v)$ divides $F_H(u, \chi_H(v)_k)$ and $F_H(u, \chi_H(v)) = (u - v)g(u, v)$ for some $g(u, v) \in H(k)[[u, v]]$. Since $F_H(u, v)_1 = u + v$ and $\chi_H(v)_1 = -v$, we conclude that $F_H(u, \chi_H(v)) = u - v + (u - v)g'(u, v) = (u - v)(1 + g'(u, v)) = (u - v)\psi(u, v)$ where $\deg(g') > 0$ and $\psi(u, v) = (1 + g'(u, v))$ a unit. □

6.7. Higher Chern classes

Let E be a vector bundle of rank r over a quasi-projective scheme X. By virtue of the projective bundle theorem, there are morphisms

$$CK_{p,q}(X) \xrightarrow{-\alpha_r} CK_{p-1,q+1}(\mathbb{P}(E)) \xrightarrow{(\oplus_{i=0}^{r-1}\alpha_i)^{-1}} \oplus_{i=0}^{r-1} CK_{p-r+i,q+r-i}(X)$$

which sends an element $a_0 \in CK_{p,q}(X)$ to $a_r \oplus \cdots \oplus a_1 \in CK_{p-r,q+r}(X) \oplus \cdots \oplus CK_{p-1,q+1}(X)$ such that $\sum_{i=0}^{r} \alpha_{r-i}(a_i) = 0$ in $CK_{p-1,q+1}(\mathbb{P}(E))$. These morphisms define the various Chern classes of E. $c_i(\mathcal{E}) : CK_{p,q}(X) \to CK_{p-i,q+i}(X)$ is given $c_i(\mathcal{E})(a_0) = a_i$ for $0 \leqslant i \leqslant r - 1$ and 0 otherwise. The total Chern class is $c(\mathcal{E}) = \oplus_{i=0}^{r} c_i(\mathcal{E})$ and the Chern polynomial is $c_t(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E})t^i$ where t is an indeterminant.

The Chern classes entertain the following properties.

1. $c_i(\mathcal{E}) = 0$ for $i > r$.
2. Push-forward. Given a proper morphism $f : X' \to X$, we have $c_i(\mathcal{E}) \circ f_* = f_* \circ c_i(f^*\mathcal{E})$.
3. Pull-back. Given $f : X' \to X$ flat of constant relative dimension, $c_i(f^*\mathcal{E}) \circ f^* = f^* \circ c_i(\mathcal{E})$.
4. Commutativity. Given E and $\mathcal{F}$ two vector bundles over X, $c_i(\mathcal{E}) \circ c_j(\mathcal{F}) = c_j(\mathcal{F}) \circ c_i(\mathcal{E})$ for all $i, j \in \mathbb{Z}$.
5. In case E is a line bundle, this definition agrees with the earlier one.
6. Whitney Sum. Given an exact sequence of vector bundles: $0 \to E' \to E \to E'' \to 0$, we have $c_t(E) = c_t(\mathcal{E}') \cdot c_t(\mathcal{E}'')$, viz., $c_k(\mathcal{E}) = \sum_{i+j=k} c_i(\mathcal{E}') \cdot c_j(\mathcal{E}'')$.

The definition of higher Chern classes and the attendant properties are formal consequences of the projective bundle theorem and the maps α_is. The details of CK-homology are not relevant. Property 1 in definitions 2, 3, 4 follows from the functoriality of $\alpha_m = (-c_1(\mathcal{O}(-1)))^m \circ p^*$.

5. We denote the old Chern class for line bundles by $\overline{c}_1$. From the definitions, $\alpha_1 = -\overline{c}_1(\mathcal{O}_{\mathbb{P}(E)}(-1))p^*$, $\alpha_0 = p^*$ and $\alpha_1 + \alpha_0 \circ c_1(\mathcal{E}) = 0$. By Property 3, $p^*c_1(\mathcal{E}) = c_1(p^*(\mathcal{E}))p^*$. Because E is a line bundle, $p : \mathbb{P}(E) \to X$ is an isomorphism. It ensues that $\mathcal{O}_{\mathbb{P}(E)}(-1) \cong \mathcal{E}$ and $(-\overline{c}_1(\mathcal{E}) + c_1(\mathcal{E})) = 0$.

6. The proof trails the course of a standard argument making use of the splitting principle. c.f. [2].

7. The connective *K*-groups

One of the most salient features of CK-homology groups is that they admit canonical transformations to K-homology groups (which include Chow groups) and algebraic K-groups. Let X be a projective scheme over k.

The $n = 2$ page long exact sequences of the BGQ spectral sequence (3) are

$$\cdots \to CK_{p-1,q+1}(X) \to CK_{p,q}(X) \to A_{p,q}(X) \to CK_{p-2,q+1}(X) \to \cdots .$$

The middle arrow is a canonical morphism $\Theta_A(X) : CK_{p,q}(X) \to A_{p,q}(X)$.

From the definition that $CK_{p,q}(X) = D^2_{p+1,q-1}(X)$, we have the canonical morphism $CK_{p,q}(X) \to K_{p+q}(\mathcal{M}_{p+1}(X)) \to K_{p+q}(\mathcal{M}(X)) = K_{p+q}(X) \cong K_{p,q}(X)$, or $\Theta_K(X) : CK_{p,q}(X) \to K_{p,q}(X)$.

The functorial properties of BGQ long exact sequences imply that Θ_A and Θ_K commute with f_*/f^* for proper/flat morphism $f : Y \to X$ respectively.

7.1. $\Theta_A(k)$ and $\Theta_K(k)$

We know from Proposition 5.1 that

$$CK_{p,q}(k) \cong \begin{cases} K_{p+q}(k), & p \geqslant 0; \\ 0, & p < 0. \end{cases} \tag{8}$$

From this, it follows, on account of (3), that

$$A_{p,q}(k) \cong \begin{cases} K_q(k), & p = 0; \\ 0, & p \neq 0. \end{cases} \tag{9}$$

The morphism $\Theta_A(k)$ is a bi-graded morphism between these groups and $\Theta_A(k)_{p,q}$ takes $CK_{p,q}(k)$ to 0 for $p > 0$ and is an isomorphism $CK_{o,q} \to A_{o,q}$.

We have seen that there is a bi-graded ring structure on $CK(k)$: $CK_{p,q}(k) \otimes CK_{r,s}(k) \to CK_{p+r,q+s}(k)$. Setting $q = s = 0$, we recover the graded ring structure on $\oplus_i K_i(k)$. Recall that the bi-graded ring structure on $K_T(k) = \oplus_{p,q\in\mathbb{Z}} K_{p,q}(k) \cong K(k)[\beta, 1/\beta]$ is given by $K_p(k)\beta^i \otimes K_q(k)\beta^j \to K_{p+q}(k)\beta^{i+j}$ and we identify $\beta \in K_{1,-1}(k)$ as the canonical generator of $K_{1,-1}(k)$. Since multiplication by the invertible element β gives an isomorphism $K_{p,q}(X) \to K_{p+1,q-1}(X)$, β is called *the Bott element* in $K_T(k)$. The functorial properties of the products imply that $\Theta_K(k)$ is a bi-graded ring homomorphism from $CK(k)$ to $K_T(k)$. Proposition 5.1 asserts that for $p \geqslant 0$, $\Theta_K(k)_{p,q} : CK_{p,q}(k) \to K_{p+q}(k) \cong K_{p+q}(k)\beta^p = K_{p,q}(k)$ is an isomorphism. We therefore have $CK(k) \cong K(k)[\beta] \hookrightarrow K_T(k) \cong K(k)[\beta, \beta^{-1}]$ via $\Theta_K(k)$. In particular, β is also identified as the canonical generator of $CK_{1,-1}(k)$.

To summarize, we have the following commutative diagram of relations among maps

$$\begin{tikzcd} A(k) \arrow[d, "\cong"] & CK(k) \arrow[l, "\Theta_A"'] \arrow[r, "\Theta_K"] \arrow[d, "\cong"] & K_T(k) \arrow[d, "="] \\ K(k) & K(k)[\beta] \arrow[l, "\pi"'] \arrow[r, "\iota"] & K(k)[\beta, 1/\beta] \end{tikzcd}$$

where ι is the natural embedding, $K(k) = \oplus_q K_q(k)$ is identified with $\oplus_q K_{0,q}(k)$ and π is the natural quotient sending β to 0.

7.2. The Bott element and the connectivity

Let X be quasi-projective over k. Observe that $CK(X)$, as a doubly graded $CK(k)$-module, can be localized at the homogeneous element $\beta \in CK_{1,-1}(k)$ and $CK(X)_\beta$ is a doubly graded $CK(k)_\beta$-module. In the last section, we affirmed that $CK(k)_\beta \cong K_T(k)$. It ensues that $\Theta_K(X)$ factors through $\Theta_K(X)_\beta : CK(X)_\beta \to K_T(X)$. On the other hand, $CK(X)/\beta CK(X)$ is a doubly graded $CK(k)/\beta CK(k) = A(k)$ module.

Theorem 7.1. *Let X be a quasi-projective scheme over k and $\beta \in CK_{1,-1}(k)$ the Bott element.*

1. $\mathrm{Im}(\Theta_A(X)_{p,q} : CK_{p,q}(X) \to A_{p,q}(X)) \cong CK_{p,q}(X)/\beta CK_{p-1,q+1}(X)$ *whereby $\Theta_A(X)$ can be identified as the natural quotient $CK(X) \to CK(X)/\beta CK(X)$. In particular, $\Theta_A(X)_{p,-p}$ is surjective and*
$$CK_{p,-p}(X)/\beta CK_{p-1,-p+1}(X) \cong CH_p(X).$$
2. *The morphism $\Theta_K(X)_\beta : CK(X)_\beta \to K_T(X)$ is an isomorphism of bi-graded $CK(k)_\beta$ or $K_T(k)$-modules whereby $\Theta_K(X)$ itself can be viewed as the canonical map $CK(X) \to CK(X)_\beta$.*

Proof. 1. By definition, $\Theta_A(X)_{p,q}$ is represented, as part of a long exact long sequence (3), by $CK_{p-1,q+1}(X) \to CK_{p,q}(X) \to A_{p,q}(X)$, so the image of $\Theta_A(X)_{p,q}$ is determined by the first arrow. It therefore suffices to show the first arrow is none other than multiplication by $\beta \in CK_{1,-1}(k)$. Regard $\mathcal{O}_k$ as a element in $\mathcal{M}_1(k)$ whose class $\beta = [\mathcal{O}_k] \in K_0(\mathcal{M}_1(k))$ generates the group. Observe the two morphisms $i : \mathcal{M}_{p-1}(X) \hookrightarrow \mathcal{M}_p(X)$ and $\times\mathcal{O}_k : \mathcal{M}_{p-1}(X) \to \mathcal{M}_p(X)$ coincide. They induce two identical maps $i_* : K_{p+q}(\mathcal{M}_{p-1}(X)) \to K_{p+q}(\mathcal{M}_p(X))$ and $\times\beta : K_{p+q}(\mathcal{M}_{p-1}(X)) \to K_{p+q}(\mathcal{M}_p(X))$ which, in turn, give rise to two equal maps $i_* : CK_{p-1,q+1}(X) \to CK_{p,q}(X)$

and $\times\beta : CK_{p-1,q+1}(X) \to CK_{p,q}(X)$ where β now is viewed as an element in $CK_{1,-1}(k)$.(c.f. Section 6.2). i_* is the map in (3) between the CK groups and we ascertain that $\operatorname{Im}\Theta_A(X)$ is isomorphic to $CK_{p,q}(X)/\beta CK_{p-1,q+1}(X)$ and under this isomorphism, $\Theta_A(X)$ is the natural quotient map $CK_{p,q}(X) \to CK_{p,q}(X)/\beta CK_{p-1,q+1}(X)$. In case $q = -p$, $p - 2 + q + 1 = -1$ and $CK_{p-2,q+1}(X) = 0$, $\Theta_A(X)$ is surjective. This completes the proof of 1.

2. Denote $CK(X)_\beta$ by $CK_\beta(X)$. We have shown in the last section that $\Theta_K(k)$ identifies $CK(k)$ with $K(k)[\beta]$ in $K_T(k) = K(k)[\beta, \beta^{-1}]$ and therefore $\Theta_K(k)_\beta$ is an isomorphism. For general X, $CK_\beta(X)_{p,q} = \sum_j CK_{p+j,q-j}(X)/\beta^j$. The $CK_{p+j,q-j}(X)$ groups form a direct system via $i_* = \times\beta$ the limit of which is $K_{p+q}(X)$ and for large j, $CK_{p+j,q-j}(X) = K_{p+q}(X)$ hence $\Theta_K(X)(CK_{p+j,q-j}(X)) = K_{p+j,q-j}(X)$. So under $\Theta_K(X)_\beta$, $(CK_{p+i,q-i}(X))/\beta^j \cong K_{p+j,q-j}(X)/\beta^j = K_{p,q}(X)$ for large j. Observe that for all j, $CK_{p+j,q-j}(X)$ has the same image in $CK_\beta(X)$ on account of the following commutative diagram

$$\begin{tikzcd}
CK_{p+j,q-j}(X) \arrow[rr, "\times\beta"] \arrow[dr, "/\beta^j"'] & & CK_{p+j+1,q-j-1}(X) \arrow[dl, "/\beta^{j+1}"] \\
& CK_\beta(X)_{p,q} &
\end{tikzcd}$$

Combining these results, we deduce that $\Theta_K(X)_\beta : CK_\beta(X)_{p,q} \to K_T(X)_{p,q}$ is an isomorphism for all integers p and q. ■

Acknowledgement

The author is immensely indebted to his advisor Alexander Merkurjev without whom this report would not have existed. Both his knowledge and insight have been an incalculable help and an unfailing source of inspiration to the author.

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